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Variational reduction for Ginzburg–Landau vortices

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Abstract

Let Ω be a bounded domain with smooth boundary in $\mathbb{R}^2$. We construct non-constant solutions to the complex-valued Ginzburg–Landau equation $\varepsilon^2\Delta u + (1 - |u|^2)u = 0$ in Ω, as $\varepsilon \to 0$, both under zero Neumann and Dirichlet boundary conditions. We reduce the problem of finding solutions having isolated zeros (vortices) with degrees ± 1 to that of finding critical points of a small C^1-perturbation of the associated renormalized energy. This reduction yields general existence results for vortex solutions. In particular, for the Neumann problem, we find that if Ω is not simply connected, then for any $k \geqslant 1$ a solution with exactly k vortices of degree one exists.

Keywords: Ginzburg–Landau vortices; Linearization; Finite-dimensional reduction

1. Introduction

We consider the Ginzburg–Landau equation

$$\varepsilon^2\Delta u + \big(1 - |u|^2\big)u = 0 \quad \text{in } \Omega, \tag{1.1}$$

where Ω is a bounded, smooth domain in $\mathbb{R}^2$, $u : \Omega \to \mathbb{C}$ and $\varepsilon > 0$ is a small parameter. (1.1) is the Euler–Lagrange equation corresponding to the *energy functional*

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$$J_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{4\varepsilon^2}\int_\Omega \left(1-|u|^2\right)^2. \tag{1.2}$$

Construction and asymptotic analysis of solutions of (1.1) and related problems as $\varepsilon \to 0$ has been a subject extensively treated in the literature during the last decade. The energy J_ε is commonly regarded as a model for the full Ginzburg–Landau energy of classical superconductivity theory [14]. It also arises in theories of superfluids and Bose–Einstein condensates. Seeking for unconstrained critical points of J_ε, namely in entire $H^1(\Omega, \mathbb{C})$, gives rise to homogeneous Neumann boundary condition for (1.1),

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \tag{1.3}$$

Bethuel et al. [3] considered Eq. (1.1) subject to a boundary condition $g : \partial\Omega \to S^1$,

$$u = g \quad \text{on } \partial\Omega, \tag{1.4}$$

with Ω star-shaped, and analyzed asymptotic behavior of families of solutions u_ε of (1.1)–(1.4), namely critical points of J_ε in the space

$$H^1_g(\Omega, \mathbb{C}) = \left\{u \in H^1(\Omega, \mathbb{C}) \mid u = g \text{ on } \partial\Omega\right\}.$$

Problem (1.1)–(1.4) corresponds to a relaxation of that of finding harmonic maps from Ω into S^1. Let us assume that g is smooth and that the degree $d = \deg(g, \partial\Omega) > 0$. It was established in [3] that for a given family of solutions u_ε there exist a number $k \geqslant 1$, and k-tuples

$$\xi = (\xi_1, \dots, \xi_k) \in \Omega^k, \qquad \mathrm{d} = (d_1, d_2, \dots, d_k) \in \mathbb{Z}^k,$$

with $\xi_i \neq \xi_j$ for all $i \neq j$ and $\sum_{j=1}^k d_j = d$, such that $u_\varepsilon(x) \to w_g(x, \xi, \mathrm{d})$ along a suitable subsequence, in $C^{1,\alpha}$-sense away from the vortices ξ_j, where

$$w_g(x, \xi, \mathrm{d}) \equiv e^{i\varphi_g(x,\xi,\mathrm{d})} \prod_{j=1}^k \left(\frac{x-\xi_j}{|x-\xi_j|}\right)^{d_j}. \tag{1.5}$$

Products in the above expression are understood in complex sense and $\varphi_g = \varphi_g(x, \xi, \mathrm{d})$ is the unique solution of the problem

$$\Delta\varphi_g = 0 \quad \text{in } \Omega, \tag{1.6}$$

$$w_g(x, \xi, \mathrm{d}) = g(x) \quad \text{on } \partial\Omega. \tag{1.7}$$

Besides, ξ must be a critical point of a *renormalized energy*, $W_g(\xi, \mathrm{d})$, characterized as the limit

$$W_g(\xi, \mathrm{d}) \equiv \lim_{\rho \to 0}\left[\int_{\Omega \setminus \bigcup_{j=1}^k B_\rho(\xi_j)} |\nabla_x w_g|^2\, dx - \pi \sum_{j=1}^k d_j^2 \log\frac{1}{\rho}\right], \tag{1.8}$$

for which explicit expression in terms of Green's functions is found in [3]. This result also holds true for general domains and families of solutions u_ε with $J_\varepsilon(u_\varepsilon) = O(\log \varepsilon)$, see [23]. In [3], accurate information was obtained for the behavior of a family of global minimizers u_ε of J_ε in $H^1_g(\Omega)$. In such a case,

$$k = d, \qquad \mathtt{d} = \mathtt{1} \equiv (1, \ldots, 1),$$

and for all small ε, u_ε possesses exactly d zeros (called vortices), each of them with degree one. Moreover, ξ is actually a global minimizer of $W_g(\cdot, \mathtt{1})$.

This result holds for general Ω as found by Struwe [35], see also [7,9]. Natural question is, of course, the reciprocal, that of finding solutions to (1.1)–(1.4) which concentrate developing vortices at other critical points of W_g. Through a heat flow method, F.H. Lin [22] found solutions which concentrate around any non-degenerate local minimum of $W_g(\cdot, \mathtt{1})$. In [7,8], non-degeneracy was lifted in the sense that if $\Lambda \subset \Omega^d$ is so that

$$\inf_{\xi \in \Lambda} W_g(\xi, \mathtt{1}) < \inf_{\xi \in \partial\Lambda} W_g(\xi, \mathtt{1}), \tag{1.9}$$

then there is a local minimizer of J_ε with exactly d vortices of degree 1 which minimize $W_g(\cdot, \mathtt{1})$ in Λ. A different proof of this fact was found by F.H. Lin and T.C. Lin in [25]. Moreover, they established existence of a solution with d vortices concentrating at any non-degenerate critical point of $W_g(\cdot, \mathtt{1})$, through heat flow analysis and topological arguments. In [24] boundary conditions yielding solutions with vortices with coexisting degrees $+1$ and -1 were found. In [1], Almeida and Bethuel devised a variational–topological approach to prove that if $d \geqslant 2$ then at least 3 solutions exist, result subsequently improved in [36] to existence of $d + 1$ solutions.

In [30] Pacard and Riviere improved the result of [24] with a completely different approach. Their construction yields very accurate information on the solution, particularly close to the zero set, and includes the case of vortex solutions with coexisting degrees 1 and -1. To state their result in more precise terms we need to introduce the standard single vortex solutions $\mathrm{w}_\pm(x)$ of respective degrees $+1$ and -1 in the plane, of the equation

$$\Delta \mathrm{w} + \big(1 - |\mathrm{w}|^2\big)\mathrm{w} = 0 \quad \text{in } \mathbb{R}^2,$$

which have the form

$$\mathrm{w}_+(x) = U(r)e^{i\theta}, \qquad \mathrm{w}_-(x) = U(r)e^{-i\theta}, \tag{1.10}$$

where (r, θ) designate usual polar coordinates $x_1 = r\cos\theta$, $x_2 = r\sin\theta$, and $U(r)$ is the unique solution of the problem

$$\begin{cases} U'' + \frac{U'}{r} - \frac{U}{r^2} + (1 - |U|^2)U = 0 & \text{in } (0, \infty), \\ U(0) = 0, \quad U(+\infty) = 1. \end{cases} \tag{1.11}$$

It is well known, see, e.g., [6] that $U'(0) > 0$ and

$$U(r) = 1 - \frac{1}{2r^2} + O\left(\frac{1}{r^4}\right) \quad \text{as } r \to +\infty.$$

Let us fix a number $k \geqslant 1$, and sets $I_\pm$ with

$$I_- \cup I_+ = \{1, \ldots, k\}, \qquad I_+ \cap I_- = \emptyset. \tag{1.12}$$

Let $\xi = (\xi_1, \ldots, \xi_k)$ be a k-tuple of distinct points of Ω, and

$$\mathtt{d} \in \{-1, 1\}^k, \quad d_j = \pm 1 \text{ if } j \in I_\pm. \tag{1.13}$$

We consider an approximation to a solution of (1.1)–(1.4) of the form

$$w_{g\varepsilon}(x, \xi, \mathtt{d}) = e^{i\varphi_g(x)} \prod_{j \in I_+} \mathrm{w}_+\left(\frac{x - \xi_j}{\varepsilon}\right) \prod_{j \in I_-} \mathrm{w}_-\left(\frac{x - \xi_j}{\varepsilon}\right), \tag{1.14}$$

where the products are understood to be equal to one if I_- or I_+ are empty. To match the degree of the boundary condition (1.4) we need

$$|I_+| - |I_-| = d, \tag{1.15}$$

with φ_g solving (1.6), (1.7) for these choices of parameters. We observe that as $\varepsilon \to 0$, $w_{g\varepsilon}$ approaches w_g given by (1.5), away from the poles.

Pacard and Riviere established in [30] that for this choice of $\mathtt{d}$, and a given non-degenerate critical point ξ_* of $W_g(\xi, \mathtt{d})$, a solution u_ε of problem (1.1)–(1.4) exists and satisfies

$$u_\varepsilon(x) = w_{g\varepsilon}(x, \xi_\varepsilon, \mathtt{d}) + o(1),$$

where $o(1) \to 0$ uniformly in Ω, and $\xi_\varepsilon \to \xi_*$.

The proof in [30] is based on a thorough analysis of the linearized operator around a canonical approximation and an application of implicit function theorem in certain classes of Hölder spaces. This approach has the advantage of being insensitive to minimizing or non-minimizing character of the critical point ξ_*, but it relies heavily on its non-degeneracy. This assumption is hard to check, except for special domains and boundary conditions. On the other hand, a topological approach to the problem of existence, like that of Almeida and Bethuel [1], gives results valid in arbitrary domains and under any boundary condition, without any non-degeneracy assumption. However, the topological analysis is difficult since it relies only on general properties of the "vortex space" and gives relatively little insight into the location or structure of the solutions found.

In problems with variational structure, higher Morse index solutions are harder to find or describe accurately through purely variational methods. This is partly the reason why less it is known for existence of vortex solutions in the Neumann problem (1.1)–(1.3). Unlike the Dirichlet problem, minimization does not produce non-trivial solutions of (1.1)–(1.3), minimizers of J_ε in $H^1(\Omega)$ are just constants with absolute value one. Worse than this, non-constant local minimizers do not exist if Ω is convex or if Ω is simply connected and ε is small, see Jimbo and Morita [18] and Serfaty [33]. If Ω is not simply connected, nonconstant local minimizers (without vortices) do exist [21] for small ε. In the full Ginzburg–Landau energy, for which natural boundary conditions are Neumann, local minimizers with vortices do exist at proper ranges of an external applied field [32].

On the other hand, a classification result similar to that in [3] is available for the Neumann problem, as found in [33]. A family of solutions u_ε to (1.1)–(1.3) with $J_\varepsilon(u_\varepsilon) = O(\log\varepsilon)$ satisfies the asymptotic behavior $u_\varepsilon(x) \to w_{\mathcal{N}}(x, \xi, \mathrm{d})$, where

$$w_{\mathcal{N}}(x, \xi, \mathrm{d}) \equiv e^{i\varphi_{\mathcal{N}}(x,\xi,\mathrm{d})} \prod_{j=1}^{k} \left(\frac{x-\xi_j}{|x-\xi_j|} \right)^{d_j}, \tag{1.16}$$

and $\varphi_{\mathcal{N}}(x, \xi, \mathrm{d})$ is the unique solution of the problem

$$\Delta\varphi_{\mathcal{N}} = 0 \quad \text{in } \Omega, \tag{1.17}$$

$$\frac{\partial\varphi_{\mathcal{N}}}{\partial\nu} = -\sum_{j=1}^{k} d_j \frac{(x-\xi_j)^{\perp}\cdot\nu}{|x-\xi_j|^2} \quad \text{on } \partial\Omega, \qquad \int_{\Omega} \varphi_{\mathcal{N}} = 0. \tag{1.18}$$

Here $x^{\perp} = (-x_2, x_1)$. A solution (1.17), (1.18) is easily seen to exist and be unique up to additive constant since mean value of the boundary condition equals zero. Like in the Dirichlet problem, ξ must be a critical point to the renormalized energy, $W_{\mathcal{N}}(\xi, \mathrm{d})$, defined as the limit

$$W_{\mathcal{N}}(\xi, \mathrm{d}) \equiv \lim_{\rho\to 0} \left[\int_{\Omega\setminus\bigcup_{j=1}^{k} B_\rho(\xi_j)} |\nabla_x w_{\mathcal{N}}|^2\, dx - \pi \sum_{j=1}^{k} d_j^2 \log\frac{1}{\rho} \right], \tag{1.19}$$

expression for which also explicit form is available, see Section 2.

In this paper we devise a method, which applies both to Dirichlet and Neumann problems, to find vortex solutions of combined degrees ± 1. This is achieved by constructing a finite-dimensional manifold of approximate solutions, parametrized by all possible locations of vortices, such that critical points of J_ε constrained to this manifold correspond to vortex solutions. Existence of critical points of this reduced functional, which turns out to be a small C^1-perturbation of the renormalized energy, can be analyzed through general topological information, without any reference to non-degeneracy. In particular this enables us to find what seem to be first general results on existence of vortex solutions in the Neumann problem, as well as new results for the Dirichlet case. This approach, sometimes called *variational reduction*, has been successfully applied in various singular perturbation elliptic problems involving *point concentration*.

Let us consider a number $k \geqslant 1$, k-tuples ξ and $\mathrm{d} \in \{-1, 1\}^k$ with corresponding sets $I_\pm$ as in (1.12), (1.13), and the associated renormalized energies W_g for the Dirichlet problem and $W_{\mathcal{N}}$ for the Neumann problem. Additionally, for the Neumann problem we define the approximation $w_{\mathcal{N}\varepsilon}(x, \xi, \mathrm{d})$ similarly as in (1.14),

$$w_{\mathcal{N}\varepsilon}(x, \xi, \mathrm{d}) = e^{i\varphi_{\mathcal{N}}(x,\xi,\mathrm{d})} \prod_{j\in I_+} \mathrm{w}_+\left(\frac{x-\xi_j}{\varepsilon}\right) \prod_{j\in I_-} \mathrm{w}_-\left(\frac{x-\xi_j}{\varepsilon}\right). \tag{1.20}$$

We say that $W_{\mathcal{N}}(\cdot, \mathrm{d})$ (respectively $W_g(\cdot, \mathrm{d})$) exhibits a *non-trivial critical point situation* in $\mathcal{D}$, open and bounded subset of Ω^k with

$$\overline{\mathcal{D}} \subset \left\{\xi \in \Omega^k \colon \xi_i \neq \xi_j, \text{ if } i \neq j\right\},$$

if there exists a $\delta > 0$ such that for any $h \in C^1(\overline{\mathcal{D}})$ with $\|h\|_{C^1(\overline{\mathcal{D}})} < \delta$, a critical point for $W_{\mathcal{N}} + h$ (respectively $W_g + h$) in $\mathcal{D}$ exists.

The following result holds.

Theorem 1.1. *Assume that $W_{\mathcal{N}}$ exhibits a non-trivial critical point situation in $\mathcal{D}$. Then there exists a solution u_ε to the Neumann problem* (1.1)–(1.3) *such that*

$$u_\varepsilon(x) = w_{\mathcal{N}\varepsilon}(x, \xi_\varepsilon, \mathrm{d}) + o(1), \tag{1.21}$$

where $o(1) \to 0$ uniformly in Ω and

$$\xi_\varepsilon \in \mathcal{D}, \qquad \nabla_\xi W_{\mathcal{N}}(\xi_\varepsilon, \mathrm{d}) \to 0. \tag{1.22}$$

The same conclusion holds for the Dirichlet problem (1.1)–(1.4) *with $W_{\mathcal{N}}$ and $w_{\mathcal{N}\varepsilon}$, respectively, replaced by W_g and $w_{g\varepsilon}$.*

In the Dirichlet problem this result lifts the non-degeneracy requirement in [25,30]. In particular it applies for the topological local minimum situation (1.9) in [9], now for combined ± 1 degrees.

We refer to a family of solutions u_ε of (1.1)–(1.3) with properties (1.21), (1.22) in some set $\mathcal{D}$ compactly contained in $\{\xi \in \Omega^k\colon\ \xi_i \neq \xi_j,\ \text{ if } i \neq j\}$, simply as a *$k$-vortex solution with degrees* d, similarly for the Dirichlet problem with $W_{\mathcal{N}}$ and $w_{\mathcal{N}\varepsilon}$, respectively, replaced by W_g and $w_{g\varepsilon}$.

As we have mentioned, this result applies to establish general results for existence of vortex solution both in Neumann and Dirichlet problems.

Theorem 1.2. *For the Neumann problem* (1.1)–(1.3)*, the following facts hold.*

(a) *A* 1*-vortex solution with degree* 1 *always exists.*
(b) *Two dipole solutions, namely two* 2*-vortex solutions with degrees* $(+1, -1)$*, always exist.*
(c) *Assume that Ω is not simply connected. Then, given any $m \geqslant 1$, there exists an m-vortex solution with degrees* $(1, \dots, 1) \in \mathbb{Z}^m$.

Formal dynamics of vortices in the simply connected case in [19,20] suggested the presence of the single-vortex and the dipole as well as their non-minimizing character. The latter fact actually follows from the results in [33], also for simply connected domains.

The rather striking presence of solutions with arbitrarily large number of vortices if the domain has non-trivial topology, is in strong analogy with a similar phenomenon found in [11] for singular limits in the Liouville equation $-\Delta u = \varepsilon^2 e^u$ under zero Dirichlet boundary condition.

As for the Dirichlet problem, we have the following results.

Theorem 1.3. *Consider the Dirichlet problem* (1.1)–(1.4) *and let* $d = \deg(g, \partial\Omega)$.

(a) *If $d = 0$ and Ω is not simply connected, then a dipole solution exists.*
(b) *If $d \geqslant 1$ then there exist at least d d-vortex solutions with degrees* $(1, \dots, 1) \in \mathbb{Z}^d$. *If Ω is not simply connected, at least $d + 1$ such solutions exist.*

The skeleton of the proofs is simple. We construct a small function $\phi(\xi)$, in such a way that critical points in ξ of $J_\varepsilon(w_{\mathcal{N}\varepsilon}(\cdot, \xi, \mathrm{d}) + \phi(\xi))$ correspond to actual critical points of J_ε. Such a procedure for Ginzburg–Landau vortices is not easy since the right functional analytic set-up to carry out this reduction is not obvious. A technical difficulty arising is the presence of slowly decaying elements in the asymptotic kernel of linearization. A difficulty of this type is present in Liouville type equations $-\Delta u = \varepsilon^2 e^u$ in two-dimensional domains, and makes construction of bubbling solutions a delicate matter, see [2,5,11,13].

It is interesting to point out that the finite-dimensional manifold represented by the functions $w_{\mathcal{N}\varepsilon}(\cdot, \xi, \mathrm{d}) + \phi(\xi)$ is nearly invariant for the associated heat flow. It is thus expected that analysis of dynamics near this manifold could potentially lead to a geometric approach to parabolic vortex dynamics in the spirit of Henry [16]. In this direction, it could provide a framework alternative to the variational one by Sandier and Serfaty in [31], and related to the multi-vortex configuration reduction by Gustafson and Sigal [15]. Energy and spectrum estimates which such a theory would require are derived in the separate work [12]. Heat flow for Ginzburg–Landau has been analyzed in [17,24,28,31].

We shall devote the rest of this paper to the proof of the above results.

2. First approximation and error estimate

In the sections to follow we will concentrate on working out the variational reduction for the Neumann problem (1.1)–(1.3). As it will become clear in the course of the arguments, just minor changes are needed for the Dirichlet problem.

Let us fix a number $k \geqslant 1$, a k-tuple $\mathrm{d} \in \{-1, 1\}^k$, a small number $\delta > 0$ and $\xi \in \Omega_\delta^k$ where

$$\Omega_\delta^k = \left\{ \xi \in \Omega^k \;\middle|\; |\xi_i - \xi_j| > 2\delta \text{ for all } i \neq j,\ \operatorname{dist}(\xi_i, \partial\Omega) > 2\delta \right\}. \tag{2.1}$$

Let $I_\pm$ be the respective sets of indices associated to ± 1 in d.

We consider the first approximation to a solution of (1.1)–(1.3) given by $w_{\mathcal{N}\varepsilon}(x, \xi, \mathrm{d})$ defined in (1.20).

The solution $\varphi_{\mathcal{N}}$ to problem (1.17), (1.18) can be decomposed as

$$\varphi_{\mathcal{N}}(x) = \sum_{j=1}^k d_j \varphi_j^*(x),$$

where

$$\Delta \varphi_j^* = 0 \quad \text{in } \Omega,$$

$$\frac{\partial \varphi_j^*}{\partial \nu} = -\frac{(x-\xi_j)^\perp \cdot \nu}{|x-\xi_j|^2} \quad \text{on } \partial\Omega, \qquad \int_\Omega \varphi_j^* = 0.$$

We observe that if $\theta(x - \xi_j)$ denotes the polar argument around the point ξ_j then we have precisely

$$\nabla \theta_j(x) = \frac{(x-\xi_j)^\perp}{|x-\xi_j|^2}.$$

Alternative way to write $w_{\mathcal{N}\varepsilon}$ is

$$w_{\mathcal{N}\varepsilon}(x) = U_0(x) \prod_{j\in I_+} e^{i(\theta_j(x)+\varphi_j^*(x))} \prod_{j\in I_-} e^{-i(\theta_j(x)+\varphi_j^*(x))},$$

where

$$U_0(x) = \prod_{j\in I_+\cup I_-} U\left(\frac{|x-\xi_j|}{\varepsilon}\right).$$

The function $w_{\mathcal{N}\varepsilon}(x)$ is intended to approximate a solution of the Neumann problem (1.1)–(1.3). Let Ω_ε denote the expanded domain $\varepsilon^{-1}\Omega$. For a function u defined in Ω let us write $v(y) = u(\varepsilon y)$, with $y \in \Omega_\varepsilon$. Then u solves (1.1)–(1.3) if and only if v satisfies

$$\begin{cases} \Delta v + (1-|v|^2)v = 0 & \text{in } \Omega_\varepsilon, \\ \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases} \tag{2.2}$$

We shall denote in what follows

$$V_0(y) = w_{\mathcal{N}\varepsilon}(\varepsilon y), \qquad \xi_j' = \frac{\xi_j}{\varepsilon} \quad \text{and} \quad \tilde{\varphi}_j^*(y) = \varphi_j^*(\varepsilon y).$$

Let us consider the approximation error of V_0 to a solution of (2.2) defined as

$$E = \Delta V_0 + \left(1-|V_0|^2\right)V_0. \tag{2.3}$$

Part of the error is, of course, how well V_0 fits the boundary condition. We set

$$F = \frac{\partial V_0}{\partial \nu}. \tag{2.4}$$

Below we shall work out estimates for E and F which are crucial for the reduction procedure.

Lemma 2.1. *There exists a constant C, depending on δ and Ω such that for all small ε and all points $\xi \in \Omega_\delta^k$ we have*

$$\sum_{j=1}^k \|E\|_{C^1(|y-\xi_j'|<3)} \leqslant C\varepsilon. \tag{2.5}$$

Moreover, we have that $E = iV_0(y)[R_1 + iR_2]$ with R_1, R_2 real-valued and

$$\left|R_1(y)\right| \leqslant C\varepsilon \sum_{j=1}^k \frac{1}{|y-\xi_j'|^3}, \qquad \left|R_2(y)\right| \leqslant C\varepsilon \sum_{j=1}^k \frac{1}{|y-\xi_j'|} \tag{2.6}$$

if $|y-\xi_j'| > 1$ for all j.

Finally, we have $F = iV_0(y)[iS_2]$ where S_2 is real-valued and

$$\|S_2\|_\infty + \varepsilon^{-1}\|\nabla S_2\|_\infty \leqslant C\varepsilon^3. \tag{2.7}$$

Proof. Let us assume first that $|y-\xi_j'| > \frac{\delta}{\varepsilon}$ for all $j \in I_+ \cup I_-$. Since the functions θ_j and φ_j^* are real-valued and since $U(r) \sim 1 - \frac{1}{2r^2}$ for large r we get

$$\left(1-|V_0|^2\right)V_0 = \left(1 - \left|\prod_j U\left(y-\xi_j'\right)\right|^2\right)V_0 = O\left(\varepsilon^2\right)V_0$$

in the considered region. On the other hand, a straightforward computation gives

$$\begin{aligned}\nabla V_0(y) = V_0(y)\Bigg\{ &\sum_{j\in I_+\cup I_-} \frac{\nabla U(|y-\xi_j'|)}{U(|y-\xi_j'|)} \\ &+ i\left[\sum_{j\in I_+}\left(\nabla\theta_j(y) + \nabla\tilde{\varphi}_j^*(y)\right) - \sum_{j\in I_-}\left(\nabla\theta_j(y) + \nabla\tilde{\varphi}_y^*(y)\right)\right]\Bigg\},\end{aligned} \tag{2.8}$$

where by slight abuse of notation we have called $\theta_j(y) = \theta(y-\xi_j')$. Taking into account that θ_j and φ_j^* are harmonic functions,

$$\begin{aligned}\Delta V_0(y) = V_0(y)\Bigg\{ &\sum_{j\in I_+\cup I_-}\left[\frac{\Delta U(|y-\xi_j'|)}{U(|y-\xi_j'|)} - \frac{|\nabla U(|y-\xi_j'|)|^2}{U^2(|y-\xi_j'|)}\right] + \left[\sum_{j\in I_+\cup I_-}\frac{\nabla U(|y-\xi_j'|)}{U(|y-\xi_j'|)}\right]^2 \\ &- \left[\sum_{j\in I_+}\left(\nabla\theta_j(y) + \nabla\tilde{\varphi}_j^*(y)\right) - \sum_{j\in I_-}\left(\nabla\theta_j(y) + \nabla\tilde{\varphi}_j^*(y)\right)\right]^2 \\ &+ 2i\left[\sum_{j\in I_+}\left(\nabla\theta_j(y) + \nabla\tilde{\varphi}_j^*(y)\right) - \sum_{j\in I_-}\left(\nabla\theta_j(y) + \nabla\tilde{\varphi}_j^*(y)\right)\right] \\ &\times\left[\sum_{j\in I_+\cup I_-}\frac{\nabla U(|y-\xi_j'|)}{U(|y-\xi_j'|)}\right]\Bigg\}.\end{aligned} \tag{2.9}$$

Now since $|\nabla\theta_j(y)| = \frac{1}{|y-\xi_j'|}$, from direct computations we get that in the region $|y-\xi_j'| > \frac{\delta}{\varepsilon}$,

$$\frac{\Delta U(|y-\xi_j'|)}{U(|y-\xi_j'|)} = O(1)\left[U''\left(\left|y-\xi_j'\right|\right) + U'\left(|y-\xi_j|\right)\left|\nabla\theta_j(y)\right|\right] = O\left(\varepsilon^2\right).$$

On the other hand, in this region one also has

$$\frac{\nabla U(|y-\xi_j'|)}{U(|y-\xi_j'|)} = O(1)\left[U'\left(\left|y-\xi_j'\right|\right)\theta_j(y)\right] = O\left(\frac{1}{|y-\xi_j'|^3}\right) = O\left(\varepsilon^3\right).$$

Furthermore, we have that $\nabla\tilde{\varphi}_j^*(y) = -\varepsilon\nabla\theta_j(y)$. Indeed, this follows from the fact that $\varphi_j^*(x) + \theta(x-\xi_j)$ is harmonic in Ω and it satisfies zero Neumann boundary conditions on $\partial\Omega$.

All this information allows us to conclude that if $|y-\xi_j'|>\frac{\delta}{\varepsilon}$ for all j then

$$\Delta V_0(y)=V_0(y)\big[O\big(\varepsilon^2\big)+i\,O\big(\varepsilon^4\big)\big],$$

so that we can conclude that in this region

$$E\equiv \Delta V_0+\big(1-|V_0|^2\big)V_0=V_0\big[O\big(\varepsilon^2\big)+i\,O\big(\varepsilon^4\big)\big]. \tag{2.10}$$

Assume now that $|y-\xi_j'|\leqslant\frac{\delta}{\varepsilon}$ for some $j\in I_+\cup I_-$. To fix idea, assume that $j\in I_+$ (the other case can be treated in the same way, except for minor changes with some signs). Concerning the nonlinear term, one gets

$$\big|V_0(y)\big|^2=U^2\big(|y-\xi_j'|\big)\bigg(\prod_{l\neq j}U^2\big(|y-\xi_l'|\big)\bigg)=U^2\big(|y-\xi_j'|\big)\big(1+O\big(\varepsilon^2\big)\big).$$

On the other hand, using again the fact that $\nabla\tilde\varphi_j^*(y)=-\varepsilon\nabla\theta_j(y)$, the linear term can be estimated as follows:

$$\begin{aligned}
\Delta V_0(y)={}&V_0(y)\bigg\{\bigg[\frac{\Delta U(|y-\xi_j'|)}{U(|y-\xi_j'|)}-\frac{|\nabla U(|y-\xi_j'|)|^2}{U^2(|y-\xi_j'|)}+O\big(\varepsilon^2\big)\bigg]\\
&+\bigg[\frac{\nabla U(|y-\xi_j'|)}{U(|y-\xi_j'|)}+O\big(\varepsilon^3\big)\bigg]^2-\big[\big(\nabla\theta_j(y)+\nabla\tilde\varphi_j^*(y)\big)+O(\varepsilon)\big]^2\\
&+2i\big[\big(\nabla\theta_j(y)+\nabla\tilde\varphi_j^*(y)\big)+O(\varepsilon)\big]\cdot\bigg[\frac{\nabla U(|y-\xi_j'|)}{U(|y-\xi_j'|)}+O\big(\varepsilon^3\big)\bigg]\bigg\}\\
={}&V_0(y)\bigg\{\frac{\Delta U(|y-\xi_j'|)}{U(|y-\xi_j'|)}-\big|\nabla\theta_j(y)\big|^2+2i\nabla\theta_j(y)\cdot\frac{\nabla U(|y-\xi_j'|)}{U(|y-\xi_j'|)}\\
&+O(\varepsilon)\nabla\theta_j(y)+O\big(\varepsilon^3\big)\frac{\nabla U(|y-\xi_j'|)}{U(|y-\xi_j'|)}+O\big(\varepsilon^2\big)\\
&+i\bigg[O(\varepsilon)\nabla\theta_j(y)\cdot\frac{\nabla U(|y-\xi_j'|)}{U(|y-\xi_j'|)}+O\big(\varepsilon^3\big)\nabla\theta_j(y)+O(\varepsilon)\frac{\nabla U(|y-\xi_j'|)}{U(|y-\xi_j'|)}\bigg]\bigg\}.
\end{aligned}$$

We thus can conclude that, for $|y-\xi_j'|\leqslant\frac{\delta}{\varepsilon}$ one has

$$\begin{aligned}
E\equiv{}&\Delta V_0+\big(1-|V_0|^2\big)V_0\\
={}&V_0\bigg(O(\varepsilon)\nabla\theta_j(y)+O\big(\varepsilon^3\big)\frac{\nabla U(|y-\xi_j'|)}{U(|y-\xi_j'|)}+O\big(\varepsilon^2\big)\\
&+i\bigg[O(\varepsilon)\nabla\theta_j(y)\frac{\nabla U(|y-\xi_j'|)}{U(|y-\xi_j'|)}+O\big(\varepsilon^3\big)\nabla\theta_j(y)+O(\varepsilon)\nabla U\big(|y-\xi_j'|\big)\bigg]\bigg).
\end{aligned}\tag{2.11}$$

From here the desired estimate (2.6) follows. Estimate (2.5) can as well be easily derived from the explicit expressions.

On the boundary of Ω_ε, we have

$$\begin{aligned}\frac{\partial V_0}{\partial \nu}(y) &= \nabla V_0(y)\cdot\nu\\ &= V_0(y)\sum_{j\in I_+\cup I_-}\frac{\nabla U(|y-\xi_j'|)}{U(|y-\xi_j'|)}\cdot\nu\\ &\quad + V_0(y)i\Bigg[\sum_{j\in I_+}\big(\nabla\theta_j(y)+\nabla\tilde\varphi_j^*(y)\big)-\sum_{j\in I_+}\big(\nabla\theta_j(y)+\nabla\tilde\varphi_j^*(y)\big)\Bigg]\cdot\nu.\end{aligned}$$

Since for $y\in\partial\Omega_\varepsilon$ we have $|y-\xi_j'|>\frac{\delta}{\varepsilon}$, we derive the estimate

$$\frac{\partial V_0}{\partial \nu}(y) = V_0(y)\big[O\big(\varepsilon^3\big)+0i\big] \tag{2.12}$$

and the L^∞ part of estimate (2.7) is thus proven. Direct differentiation completes the proof. In order to prove the second part of estimate (2.7), we observe that

$$\nabla S_2(y) = \nabla\Bigg(\sum_{j\in I_+\cup I_-}\frac{\nabla U(|y-\xi_j'|)}{U(|y-\xi_j'|)}\cdot\nu\Bigg).$$

Using again the estimates proved above, we get that, for $y\in\partial\Omega_\varepsilon$,

$$\big\|\nabla S_2(y)\big\|_\infty\leqslant C\varepsilon^4.$$

This completes the proof. □

3. Formulation of the problem

We shall look for a solution of problem (2.2) in the form of a small perturbation of V_0. There are different ways to write such a perturbation. Since we have a "small" error, as described in the previous lemma, the equation for the perturbation is a linear one with a right-hand side given by this error perturbed by a lower order nonlinear term. The mapping properties of this linear operator are, of course, fundamental in solving for such a perturbation. Not only this, the nonlinearity must truly remain small if, say, an iteration scheme is produced. A characteristic of Ginzburg–Landau not present in other singular perturbation problems is its great sensitivity to the way the perturbation is written, since good mapping properties are not at all indifferent to the way the nonlinearity is expressed. An obvious way to write this perturbation is an additive way, say $v=V_0+\phi$. The nonlinearity produced when substituting this ansatz in (2.2) is a polynomial in ϕ carrying quadratic and cubic terms. While this looks good in principle, it turns out to be rather disastrous for any reasonable mapping theory one develops for the linear operator that appears. Another way to express such a perturbation is $v=V_0e^{i\psi}$ with small ψ. As we will see, this expression adapts very well to the equation, but it is not too good near them: not all functions close to V_0 can be written in this form since this expression for v would leave the zero (vortex) set invariant. It turns out to be of great convenience to consider instead an ansatz that combines the additive one near the vortices with the multiplicative one. This is the way in which we formulate the problem next.

Let $\tilde{\eta}:\mathbb{R}\to\mathbb{R}$ be a smooth cut-off function that $\tilde{\eta}(s)=1$ for $s\leqslant 1$ and $\tilde{\eta}(s)=0$ for $s\geqslant 2$. Define $\eta(y)$ to be the function

$$\eta(y)=\sum_{j\in I_+\cup I_-}\tilde{\eta}\big(|y-\xi'_j|\big).$$

We shall look for solution of (2.2) of the form

$$v(y)=\eta(V_0+iV_0\psi)+(1-\eta)V_0e^{i\psi}, \tag{3.1}$$

where ψ is small, however, possibly unbounded near the vortices. We write $\psi=\psi_1+i\psi_2$, with ψ_1 and ψ_2 real-valued. Setting

$$\phi=iV_0\psi, \tag{3.2}$$

we shall, however, require that ϕ is bounded (and smooth) near the vortices.

The ansatz (3.1) is additive, $v=V_0+\phi$, close to the vortices ξ'_j and multiplicative as soon as y is at distance greater than 2 from them. In terms of ϕ the ansatz takes the form

$$v(y)=\eta(V_0+\phi)+(1-\eta)V_0e^{\phi/V_0}. \tag{3.3}$$

Let us observe that

$$v=V_0+iV_0\psi+(1-\eta)V_0\big[e^{i\psi}-1-i\psi\big].$$

Let us denote

$$\gamma_1(y)=(1-\eta)V_0\big[e^{i\psi}-1-i\psi\big], \tag{3.4}$$

function supported in the set $\{y\in\Omega_\varepsilon\colon\ |y-\xi'_j|>1 \text{ for all } j\}$.

A direct computation shows that v is a solution to the equation in (2.2) of the form (3.1) if and only if

$$\mathcal{L}^\varepsilon(\psi)=R+N(\psi)\quad \text{in } \Omega_\varepsilon, \tag{3.5}$$

$$\frac{\partial\psi}{\partial\nu}=S\quad \text{on } \partial\Omega_\varepsilon, \tag{3.6}$$

where

$$R=iV_0^{-1}E,\qquad S=iV_0^{-1}F \tag{3.7}$$

with E, F the error terms given by (2.3) and (2.4), $\mathcal{L}^\varepsilon(\psi)$ is the linear operator defined by

$$\mathcal{L}^\varepsilon(\psi)=\Delta\psi+2\frac{\nabla V_0}{V_0}\cdot\nabla\psi-2i|V_0|^2\psi_2+\eta\frac{E}{V_0}\psi \tag{3.8}$$

and $N(\psi)$ is the nonlinear operator in ψ defined by

$$N(\psi) = iV_0^{-1}\big[\Delta\gamma_1 + \big(1 - |V_0|^2\big)\gamma_1 - 2\,\mathrm{Re}(\bar{V}_0 iV_0\psi)(iV_0\psi + \gamma_1) \\ - \big(2\,\mathrm{Re}(\bar{V}_0\gamma_1) + |iV_0\psi + \gamma_1|^2\big)(V_0 + iV_0\psi + \gamma_1)\big] + (\eta - 1)\frac{E}{V_0}\psi,$$

where γ_1 is defined by (3.4). Directly from the form of the ansatz, we see that, in the region $|y - \xi_j'| > 2$ for all j, Eq. (3.5) takes the simple form

$$\mathcal{L}^\varepsilon(\psi) = R - i(\nabla\psi)^2 + i|V_0|^2\big(1 - e^{-2\psi_2} + 2\psi_2\big). \tag{3.9}$$

We intend next to describe in more accurate form the equation above. Let us fix an index $1 \leqslant j \leqslant k$ and let us define α_j by the relation

$$V_0(y) = \mathrm{w}\big(y - \xi_j'\big)\alpha_j(y), \tag{3.10}$$

where by w we mean w_+ or w_- depending whether $j \in I_+$ or $j \in I_-$, in other words

$$\alpha_j(y) = e^{i\varphi_{\mathcal{N}}(\varepsilon y)} \prod_{l \neq j} \mathrm{w}\big(y - \xi_l'\big). \tag{3.11}$$

For $|y - \xi_j'| < \frac{\delta}{\varepsilon}$, there are two real functions A_j and B_j so that

$$\alpha_j(y) = e^{iA_j(y) + B_j(y)}; \tag{3.12}$$

furthermore, a direct computation shows that, in this region, one has

$$\nabla A_j(y) = O(\varepsilon), \qquad \Delta A_j(y) = O\big(\varepsilon^2\big) \tag{3.13}$$

and

$$\nabla B_j(y) = O\big(\varepsilon^3\big), \qquad \Delta B_j(y) = O\big(\varepsilon^4\big). \tag{3.14}$$

Observe that the estimates (3.13) and (3.14) above hold true in any region of points at a distance greater than $\frac{\delta}{\varepsilon}$ from any ξ_l', with $l \neq j$.

Recall that $\psi = \psi_1 + i\psi_2$ with ψ_1, ψ_2 real-valued. Then Eq. (3.9) for $|y - \xi_j'| > 2$ becomes

$$\Delta\psi_1 + 2\left(\nabla B_j + \frac{U'(|y - \xi_j'|)}{U(|y - \xi_j'|)}\frac{y - \xi_j'}{|y - \xi_j'|}\right)\cdot\nabla\psi_1 \\ - 2\big(\nabla A_j + \nabla\theta_j(y)\big)\cdot\nabla\psi_2 + 2\nabla\psi_1\nabla\psi_2 - R_1 = 0, \tag{3.15}$$

and

$$\Delta\psi_2 - 2|V_0|^2\psi_2 + 2\left(\nabla B_j + \frac{U'(|y - \xi_j'|)}{U(|y - \xi_j|)}\frac{y - \xi_j'}{|y - \xi_j'|}\right)\cdot\nabla\psi_2 + 2\big(\nabla A_j + \nabla\theta_j(y)\big)\cdot\nabla\psi_1 \\ + |V_0|^2\big(e^{-2\psi_2} - 1 - 2\psi_2\big) + |\nabla\psi_1|^2 - |\psi_2|^2 - R_2 = 0. \tag{3.16}$$

Next we shall write the equation of problem (2.2) in terms of the function ϕ defined in (3.2) for $|y-\xi_j'|<\frac{\delta}{\varepsilon}$. It is more convenient to do this in the translated variable $z=y-\xi_j'$. We define the function $\phi_j(z)$ through the relation

$$\phi_j(z)=i\mathrm{w}(z)\psi(y), \quad |z|<\frac{\delta}{\varepsilon}, \tag{3.17}$$

with $y=\xi_j'+z$ namely

$$\phi(y)=\phi_j(z)\alpha_j(z),$$

where, with abuse of notation, we write $\alpha_j(z)$ to mean the function $\alpha_j(y)$ defined in (3.10) and ϕ in (3.2). Hence, in the translated variable, the ansatz (3.3) becomes in this region

$$v(y)=\alpha_j(z)\left(\mathrm{w}(z)+\phi_j(z)+\big(1-\tilde{\eta}(z)\big)\mathrm{w}(z)\left[e^{\phi_j(z)/\mathrm{w}(z)}-1-\frac{\phi_j(z)}{\mathrm{w}(z)}\right]\right). \tag{3.18}$$

Let us call $\gamma_j=(1-\tilde{\eta})\mathrm{w}[e^{\phi_j/\mathrm{w}}-1-\phi_j/\mathrm{w}]$. The support of this function is contained in the set $|z|>1$. Let us consider the operator L_j^ε defined in the following way: for ϕ_j, ψ linked through formula (3.17) we set

$$L_j^\varepsilon(\phi_j)(z)=i\mathrm{w}(z)\mathcal{L}^\varepsilon(\psi)\big(\xi_j'+z\big). \tag{3.19}$$

Then another way to say that v solves (2.2) in the region $|y-\xi_j'|<\frac{\delta}{\varepsilon}$ is

$$L_j^\varepsilon(\phi_j)=\tilde{R}_j+\tilde{N}_j(\phi_j), \tag{3.20}$$

where explicitly L_j^ε becomes

$$\begin{aligned} L_j^\varepsilon(\phi_j)={}&L^0(\phi_j)+2\big(1-|\alpha_j|^2\big)\operatorname{Re}(\bar{\mathrm{w}}\phi_j)\mathrm{w}+2\frac{\nabla\alpha_j}{\alpha_j}\cdot\nabla\phi_j\\ &+2i\phi_j\frac{\nabla\alpha_j}{\alpha_j}\cdot\frac{\nabla\mathrm{w}}{\mathrm{w}}+\tilde{\eta}\frac{E_j}{V_0^j}\phi_j, \end{aligned} \tag{3.21}$$

where L^0 is the linear operator defined by

$$L^0(\phi)=\Delta\phi+\big(1-|\mathrm{w}|^2\big)\phi-2\operatorname{Re}(\bar{\mathrm{w}}\phi)\mathrm{w}, \tag{3.22}$$

E_j is given by

$$E_j=\Delta V_0^j+\big(1-\big|V_0^j\big|^2\big)V_0^j,$$

where V_0^j is the function V_0 translated to ξ_j', namely $V_0^j(z)=V_0(z+\xi_j')$. Observe that, in terms of α_j, E_j takes the expression

$$E_j=2\nabla\alpha_j\cdot\nabla\mathrm{w}+\mathrm{w}\Delta\alpha_j+\big(1-|\alpha_j|^2\big)|\mathrm{w}|^2\alpha_j\mathrm{w}. \tag{3.23}$$

The term $\tilde{R}_j$ in (3.20) is

$$\tilde{R}_j = -\alpha_j^{-1} E_j \tag{3.24}$$

while the nonlinear term $\tilde{N}_j(\phi_j)$ is given by

$$\begin{aligned}\tilde{N}_j(\phi_j) = -\bigg[&\frac{\Delta(\alpha_j\gamma_j)}{\alpha_j} + \big(1-\big|V_0^j\big|^2\big)\gamma_j - 2|\alpha_j|^2\operatorname{Re}(\bar{\mathrm{w}}\phi_j)(\phi_j+\gamma_j)\\ &- \big(2|\alpha_j|^2\operatorname{Re}(\bar{\mathrm{w}}\gamma_j) + |\alpha_j|^2|\phi_j+\gamma_j|^2\big)(\mathrm{w}+\phi_j+\gamma_j)\bigg] + (\tilde{\eta}-1)\frac{E_j}{V_0^j}\phi_j.\end{aligned} \tag{3.25}$$

Taking into account the explicit form of the function α_j we get

$$\nabla\alpha_j(z) = O(\varepsilon), \qquad \Delta\alpha_j(z) = O\big(\varepsilon^2\big), \qquad \big|\alpha_j(z)\big| \sim 1 + O\big(\varepsilon^2\big) \tag{3.26}$$

provided that $|y - \xi_j'| < \frac{\delta}{\varepsilon}$. With this in mind, we see that the linear operator L_j^ε is a small perturbation of L^0.

We intend to solve problem (3.5), (3.6). To do so, we need to analyze the possibility to invert the operator $\mathcal{L}^\varepsilon$ in order to express the equation as a fixed point problem. It is not expected this operator to be in general invertible. Indeed, its version L_j^ε in the ϕ_j-variable is a small perturbation of the operator L^0 in (3.22). When regarded in entire $\mathbb{R}^2$ this operator does have a kernel: functions w_{x_l} and $i\mathrm{w}$ annihilate it. In suitable spaces (for instance L^∞), these functions are known to span the entire kernel, see [26,30]. In a suitable "orthogonal" to this kernel, the bilinear form associated to this operator turns out to be uniformly positive definite, a main fact we shall use in our construction in a form established in [10]. Sections 4, 5 are intended to solve a suitably projected version of problem (3.5), (3.6), for which a linear theory is in order, after which the resolution comes from a direct application of contraction mapping principle. The next step is to adjust the points ξ in order to have a solution to the full problem. The latter problem turns out to be equivalent to a variational problem in ξ which we analyze in Sections 6, 7. The theorems will be a consequence of solving this finite-dimensional problem in different situations. We do this in Section 8.

4. Projected linear theory for $\mathcal{L}^\varepsilon$

Let us consider a small, fixed number $\delta > 0$, and points $\xi \in \Omega_\delta^k$, the set defined in (2.1). We also call $\xi_j' = \xi_j/\varepsilon$. We consider first the following linear problem:

$$\mathcal{L}^\varepsilon(\psi) = h + c_0\varepsilon^2\chi_{\Omega_\varepsilon\setminus\bigcup_{j=1}^k B(\xi_j',\delta/\varepsilon)} \quad \text{in } \Omega_\varepsilon, \tag{4.1}$$

$$\frac{\partial\psi}{\partial\nu} = g \quad \text{on } \partial\Omega_\varepsilon, \tag{4.2}$$

$$\int\limits_{\Omega_\varepsilon\setminus\bigcup_{j=1}^k B(\xi_j',\delta/\varepsilon)} \psi_1 = 0, \qquad \operatorname{Re}\int\limits_{|z|<1} \bar{\phi}_j\mathrm{w}_{x_l} = 0, \quad \text{for all } j, l. \tag{4.3}$$

The operator $\mathcal{L}^\varepsilon$ is given by (3.8), ψ_1 denotes the real part of ψ and ϕ_j is the function defined from ψ by relation (3.17). Here and in what follows we denote by χ_A the function defined as

$$\chi_A(y) = 1 \quad \text{if } y \in A, \qquad \chi_A(y) = 0, \quad \text{otherwise.}$$

We will establish a priori estimates for this problem. To this end we shall conveniently introduce adapted norms. Let us fix numbers $0 < \gamma, \sigma < 1$, denote $r_j = |y - \xi_j'|$ and define

$$\begin{aligned}\|\psi\|_* = {} & \sum_{j=1}^k \|\phi_j\|_{C^{2,\gamma}(|z|<2)} + \sum_{j=1}^k \|\phi_j\|_{C^{1,\gamma}(|z|<3)} \\ & + \sum_{j=1}^k \big[\|\psi_1\|_{L^\infty(r_j>2)} + \|r_j \nabla \psi_1\|_{L^\infty(r_j>2)}\big] \\ & + \sum_{j=1}^k \big[\big\|r_j^{1+\sigma}\psi_2\big\|_{L^\infty(r_j>2)} + \big\|r_j^{1+\sigma}\nabla\psi_2\big\|_{L^\infty(r_j>2)}\big],\end{aligned} \tag{4.4}$$

$$\|h\|_{**} = \sum_{j=1}^k \|\tilde{h}_j\|_{C^{0,\gamma}(|z|<3)} + \sum_{j=1}^k \big[\big\|r_j^{2+\sigma}h_1\big\|_{L^\infty(r_j>2)} + \big\|r_j^{1+\sigma}h_2\big\|_{L^\infty(r_j>2)}\big]. \tag{4.5}$$

Here we have denoted $\tilde{h}_j(z) = i\mathrm{w}(z)h(z+\xi_j')$. Besides, we define

$$\begin{aligned}\|g\|_{***} = {} & \varepsilon^{-1}\|g_1\|_{L^\infty(\partial\Omega_\varepsilon)} + \varepsilon^{-2}\|\nabla g_1\|_{L^\infty(\partial\Omega_\varepsilon)} \\ & + \varepsilon^{-1-\sigma}\|g_2\|_{L^\infty(\partial\Omega_\varepsilon)} + \varepsilon^{-2-\sigma}\|\nabla g_2\|_{L^\infty(\partial\Omega_\varepsilon)}.\end{aligned} \tag{4.6}$$

We want to prove the following result.

Lemma 4.1. *There exists a constant $C > 0$, dependent on δ and Ω but independent of c_0, such that for all ε sufficiently small, all points $\xi \in \Omega_\delta^k$ and any solution of problem* (4.1)–(4.3) *we have*

$$\|\psi\|_* \leqslant C\big[|\log\varepsilon|\|h\|_{**} + \|g\|_{***}\big]. \tag{4.7}$$

Proof. We argue by contradiction. Let us assume the existence of sequences $\varepsilon = \varepsilon_n \to 0$, points $\xi_{nj} \to \xi_j^* \in \Omega$ with $\xi_j^* \neq \xi_i^*$ for all $i \neq j$, and functions ψ^n, h_n, g_n which satisfy

$$\mathcal{L}_n^\varepsilon(\psi^n) = h_n + c_n\varepsilon_n^2\chi_{\Omega_{\varepsilon_n}\setminus\bigcup_{j=1}^k B(\xi_{nj}',\delta/\varepsilon_n)} \quad \text{in } \Omega_{\varepsilon_n},$$

$$\frac{\partial\psi^n}{\partial\nu} = g_n \quad \text{on } \partial\Omega_{\varepsilon_n},$$

$$\int_{\Omega\setminus\bigcup_{j=1}^k B(\xi_{nj}',\delta/\varepsilon_n)} \psi_1^n = 0, \qquad \mathrm{Re}\int_{|y|<1} \bar{\phi}_j^n \mathrm{w}_{x_l} = 0, \quad \forall j, l,$$

with

$$\left\|\psi^n\right\|_* = 1, \qquad |\log\varepsilon_n| \|h_n\|_{**} + \|g_n\|_{***} \to 0.$$

As a first step we shall show that the sequence of numbers c_n is bounded. We observe from (3.8) that on $|y - \xi'_{nj}| > \delta/\varepsilon_n$ for all j

$$\mathrm{Re}\big(\mathcal{L}_n^\varepsilon\big(\psi^n\big)\big) = \Delta\psi_1^n + O\big(\varepsilon_n^3\big)\nabla\psi_1^n + O(\varepsilon_n)\nabla\psi_2^n,$$

and hence, integrating on $\Omega_{\varepsilon_n} \setminus \bigcup_{j=1}^k B(\xi'_{nj}, \delta/\varepsilon_n)$, we get the estimate

$$|c_n| \leqslant C\left| \int\limits_{\bigcup_{j=1}^k \partial B(\xi'_{nj},\delta/\varepsilon_n)} \frac{\partial\psi_1^n}{\partial\nu} - \int\limits_{\partial\Omega_{\varepsilon_n}} \frac{\partial\psi_1^n}{\partial\nu}\right| + C\varepsilon_n^\sigma\big(\left\|\psi^n\right\|_* + \|h_n\|_{**}\big)$$

so that

$$|c_n| \leqslant C\big(\left\|\psi^n\right\|_* + \|g_n\|_{***} + \varepsilon_n^\sigma\|h_n\|_{**}\big).$$

It follows that c_n is bounded. We assume then that $c_n \to c_*$. Next we will find that actually $c_* = 0$ and that ψ^n approaches zero. Let us set $\tilde\psi^n(x) = \psi^n(x/\varepsilon_n)$. It is directly checked, from the bounds assumed, that given any number $\delta' > 0$ we have

$$\Delta\tilde\psi_1^n = O\big(\varepsilon_n^\sigma\big) + c_n\chi_{\Omega_{\varepsilon_n}\setminus\bigcup_{j=1}^k B(\xi_{nj},\delta)} \quad \text{in } \Omega \Big\backslash \bigcup_{j=1}^k B\big(\xi_j^n, \delta'\big),$$

$$\frac{\partial\tilde\psi_1^n}{\partial\nu} = o(1) \quad \text{on } \partial\Omega,$$

and, moreover,

$$\left\|\tilde\psi_1^n\right\|_\infty \leqslant 1, \qquad \left\|\nabla\tilde\psi_1^n\right\|_\infty \leqslant C_{\delta'}.$$

Passing to a subsequence, we then get that $\tilde\psi_1^n$ converges uniformly over compact subsets of $\Omega \setminus \{\xi_1^*, \dots, \xi_k^*\}$ to a function $\tilde\psi_1^*$ with $|\tilde\psi_1^*| \leqslant 1$ which solves

$$\Delta\tilde\psi_1^* = c_*\chi_{\Omega\setminus\bigcup_{j=1}^k B(\xi_j^*,\delta)} \quad \text{in } \Omega,$$

$$\frac{\partial\tilde\psi_1^*}{\partial\nu} = 0 \quad \text{on } \partial\Omega.$$

The above relation clearly implies that $c_* = 0$, and hence that $\tilde\psi_1^*$ is constant. But the orthogonality condition for $\tilde\psi_1^n$ passes to the limit and this constant must be zero. It follows that $\tilde\psi_1^n$ goes to zero uniformly and in C^1-sense away from the points $\xi_1^*, \dots, \xi_k^*$. This implies in particular that

$$\left|\psi_1^n\right| + \varepsilon_n^{-1}\left|\nabla\psi_1^n\right| \to 0 \quad \text{on } \left|z - \xi'_{jn}\right| \geqslant \frac{\delta}{2\varepsilon_n},$$

uniformly. Let us now consider the imaginary part of the equation. From (3.8) we argue that

$$-\Delta\psi_2^n + 2\psi_2^n = o\big(\varepsilon_n^{1+\sigma}\big) \quad \text{in } \Omega_{\varepsilon_n} \setminus \bigcup_{j=1}^{k} B\left(\xi'_{nj}, \frac{\delta}{2\varepsilon_n}\right),$$

$$\frac{\partial\tilde{\psi}_2^n}{\partial\nu} = o\big(\varepsilon_n^{1+\sigma}\big) \quad \text{on } \partial\Omega_{\varepsilon_n},$$

while globally in this region $\psi_2^n = O(\varepsilon_n^{1+\sigma})$. A suitable use of barriers yields then that actually $\psi_2^n = o(\varepsilon_n^{1+\sigma})$ in the C^1-sense, always in this region. Let us consider now a smooth cut-off function $\hat{\eta}$ with $\hat{\eta}(s) = 1$ if $s < \frac{1}{2}$, $\hat{\eta}(s) = 0$ if $s > 1$, and define

$$\hat{\psi}^n(z) = \hat{\eta}\left(\frac{\varepsilon_n|z - \xi'_{nj}|}{\delta}\right)\psi^n(z).$$

Let us compute the equation satisfied by $\hat{\psi}^n$. We observe that, for real and imaginary parts we get the estimates

$$\nabla_z\hat{\eta}\nabla\psi^n = \begin{bmatrix} o(\varepsilon_n^2) \\ O(\varepsilon_n^{2+\sigma}) \end{bmatrix}, \qquad \psi^n\Delta_z\hat{\eta} = \begin{bmatrix} o(\varepsilon_n^2) \\ O(\varepsilon_n^{2+\sigma}) \end{bmatrix}, \qquad \frac{\nabla V_0}{V_0}\nabla_z\hat{\eta} = \begin{bmatrix} o(\varepsilon_n^4) \\ O(\varepsilon_n^{3+\sigma}) \end{bmatrix}.$$

Thus we get

$$\mathcal{L}_n^{\varepsilon}\big(\hat{\psi}^n\big) = o(1)\begin{bmatrix} \frac{1}{r_j^{2+\sigma}} + \varepsilon_n^2 \\ \frac{1}{r_j^{1+\sigma}} \end{bmatrix} \quad \text{in } B\left(\xi'_{nj}, \frac{\delta}{\varepsilon_n}\right),$$

$$\hat{\psi}^n = 0 \quad \text{on } \partial B\left(\xi'_{nj}, \frac{\delta}{\varepsilon_n}\right). \tag{4.8}$$

The following intermediate result provides an outer estimate. For notational simplicity we shall omit the subscript n in the quantities involved.

Lemma 4.2. *There exist positive numbers R_0, C such that for all large n*

$$\begin{aligned}&\|\hat{\psi}_1\|_{L^\infty(r_j>R_0)} + \|r_j\nabla\hat{\psi}_1\|_{L^\infty(r_j>R_0)} + \|r_j^{1+\sigma}\hat{\psi}_2\|_{L^\infty(r_j>R_0)} + \|r_j^{1+\sigma}\nabla\hat{\psi}_2\|_{L^\infty(r_j>R_0)} \\ &\quad \leqslant C\big[\|\hat{\phi}\|_{L^\infty(r_j<2R_0)} + o(1)\big],\end{aligned}$$

where $\hat{\phi} = iV_0\hat{\psi}$.

Proof. From (3.15) it is directly checked that the following relations hold for $r_j > 2$

$$-\Delta\hat{\psi}_1 = O\left(\frac{1}{r_j^3}\right)\nabla\hat{\psi}_1 + O\left(\frac{1}{r_j}\right)\nabla\hat{\psi}_2 + o(1)\left(\frac{1}{r_j^{2+\sigma}} + \varepsilon^2\right), \tag{4.9}$$

$$-\Delta\hat{\psi}_2 + 2|\alpha_j|^2|\mathrm{w}_j|^2\hat{\psi}_2 + O\left(\frac{1}{r_j^3}\right)\nabla\hat{\psi}_2 = O\left(\frac{1}{r_j}\right)\nabla\hat{\psi}_1 + o(1)\,\frac{1}{r_j^{1+\sigma}}, \tag{4.10}$$

where α_j is given by (3.10) and $\mathrm{w}_j(y) = \mathrm{w}(y - \xi_j')$. Let us call p_1, p_2 the respective right-hand sides of Eqs. (4.9) and (4.10). Then we see that

$$|p_2| \leqslant \frac{B}{r_j^{1+\sigma}}, \quad B = \left\|r_j^\sigma \nabla\hat{\psi}_1\right\|_{L^\infty(r_j>1)} + o(1).$$

The use of a barrier and elliptic estimates then yield

$$\left|\nabla\hat{\psi}_2\right| + \left|\hat{\psi}_2\right| \leqslant \frac{B + \|\hat{\psi}_2\|_{L^\infty(r_j=2)}}{r_j^{1+\sigma}}, \quad 2 < r_j < \frac{\delta}{\varepsilon}.$$

Let us use these estimates to now estimate p_1. We get

$$|p_1| \leqslant \frac{C}{r_j^{2+\sigma}}\left[\left\|\nabla\hat{\psi}_1\right\|_{L^\infty(r_j>1)} + \left\|r_j^{1+\sigma}\nabla\hat{\psi}_2\right\|_{L^\infty(r_j>1)} + o(1)\right] + o\left(\varepsilon^2\right),$$

and hence

$$|p_1| \leqslant \frac{B'}{r_j^{2+\sigma}} + o\left(\varepsilon^2\right), \quad B' = C\left[\left\|r_j^\sigma\nabla\hat{\psi}_1\right\|_{L^\infty(r_j>1)} + \left\|\hat{\psi}_2\right\|_{L^\infty(r_j=2)} + o(1)\right].$$

It is easy to see that a supersolution for Eq. (4.9) is given by

$$\omega(z) = \frac{B'}{\sigma^2}\left(1 - \frac{1}{r_j^\sigma}\right) + o(1)\left(\delta^2 - r_j^2\varepsilon^2\right) + \left\|\hat{\psi}_1\right\|_{L^\infty(r_j=1)}$$

and hence

$$\left\|\hat{\psi}_1\right\|_{L^\infty(r_j>1)} \leqslant C + \left\|\hat{\psi}_1\right\|_{L^\infty(r_j=1)}.$$

Now we seek for an estimate for $\nabla\hat{\psi}_1$. Let us define $\tilde{\psi}_1(z) = \hat{\psi}_1(\xi_j + R(e+z))$ where $|e| = 1$ and $R < \frac{\delta}{\varepsilon}$. Then for $|z| \leqslant \frac{1}{2}$ we have

$$\left|\Delta\tilde{\psi}_1\right| \leqslant CB' + o(1).$$

Since, also, $|\tilde{\psi}_1| \leqslant CB'$ in this region, it follows from elliptic estimates that $|\nabla\tilde{\psi}_1(0)| \leqslant CB'$. Since R and e are arbitrary, what we have established is

$$\left|\hat{\psi}_1\right| + \left|r_j\nabla\hat{\psi}_1\right| \leqslant C\left[\left\|r_j^\sigma\nabla\hat{\psi}_1\right\|_{L^\infty(r_j>1)} + \left\|\hat{\psi}_1\right\|_{L^\infty(1<r_j<2)} + o(1)\right].$$

Now,

$$\left\|r_j^\sigma\nabla\hat{\psi}_1\right\|_{L^\infty(r_j>1)} \leqslant R_0^\sigma\left\|\nabla\hat{\psi}_1\right\|_{L^\infty(1<r_j<R_0)} + \frac{1}{R_0^{1-\sigma}}\left\|r_j\nabla\hat{\psi}_1\right\|_{L^\infty(r_j>R_0)},$$

thus fixing R_0 sufficiently large we obtain

$$\left|\hat{\psi}_1\right| + \left|r_j \nabla \hat{\psi}_1\right| \leqslant C\left[\left\|\hat{\psi}_1\right\|_{C^1(1<r_j<R_0)} + o(1)\right] \quad \text{for } r_j > 2,$$

and also

$$\left|\hat{\psi}_2\right| + \left|\nabla \hat{\psi}_2\right| \leqslant \frac{C}{r_j^{1+\sigma}}\left[\|\hat{\psi}\|_{C^1(1<r_j<R_0)} + o(1)\right] \quad \text{for } r_j > 2.$$

The lemma is proven. □

Continuation of the proof of Lemma 4.1. Let us go back to the contradiction argument. Since $\|\psi\|_* = 1$, and the corresponding portion of this norm of ψ goes to zero on the region $r_j > \delta'/\varepsilon$ for all j, for any given $\delta' > 0$, we conclude from the previous lemma that necessarily, for some index j and $m > 0$ we have

$$\left\|\hat{\phi}_j\right\|_{C^{2,\gamma}(|z|<R_0)} \geqslant m, \tag{4.11}$$

where, as in (3.17),

$$\hat{\phi}_j(z) = i\mathrm{w}(z)\hat{\psi}\left(\xi_j' + z\right).$$

Let us consider the decomposition

$$\hat{\psi}\left(\xi_j' + z\right) = \hat{\psi}^0(r) + \hat{\psi}^\perp(z), \quad r = |z|,$$

$$\hat{\psi}^0(r) = \frac{1}{2\pi r} \int_{|z|=r} \hat{\psi}\left(\xi_j' + z\right) d\sigma(z),$$

and correspondingly write

$$\hat{\phi}_j = \hat{\phi}^0 + \hat{\phi}^\perp, \qquad \hat{\phi}^0 = i\mathrm{w}\hat{\psi}^0, \qquad \hat{\phi}^\perp = i\mathrm{w}\hat{\psi}^\perp. \tag{4.12}$$

From Eq. (4.8) and analyzing the remaining terms, we see that

$$\begin{aligned} L^0\left(\hat{\phi}_j\right) &= G \quad \text{in } B(0, \delta/\varepsilon), \\ \hat{\phi}_j &= 0 \quad \text{on } \partial B(0, \delta/\varepsilon), \end{aligned} \tag{4.13}$$

where $G = o(1/|\log \varepsilon|)$ for $r < 2$ and

$$H = i\mathrm{w}^{-1} G = o(1) \begin{bmatrix} \frac{1}{|\log \varepsilon| r^{2+\sigma}} + \varepsilon^2 \\ \frac{1}{|\log \varepsilon| r^{1+\sigma}} \end{bmatrix} \quad \text{for } r > 1.$$

Let us decompose $G = G^0 + G^\perp$ in analogous way to (4.12). We directly check that

$$L^0\left(\hat{\phi}^\perp\right) = G^\perp \quad \text{in } B(0, \delta/\varepsilon),$$

$$\hat{\phi}^{\perp} = 0 \quad \text{on } \partial B(0, \delta/\varepsilon).$$

From this estimate and the fact that $\|\hat{\psi}\|_*$ is uniformly bounded we find that

$$\operatorname{Re} \int\limits_{B(0,\delta/\varepsilon)} \bar{G}^{\perp} \hat{\phi}^{\perp} = o(1).$$

Define $B(\phi, \phi) = \operatorname{Re} \int_{B(0, \frac{\delta}{\varepsilon})} L^0(\phi)\bar{\phi}$. From the result in Lemma A.1 in Appendix A, it follows that there exists a number $\alpha > 0$ such that

$$\alpha \Biggl\{ \int\limits_{B(0,\delta/\varepsilon)} \frac{|\phi^{\perp}|^2}{1+r^2} + \int\limits_{B(0,\delta/\varepsilon)} \bigl|\operatorname{Re}\bigl(\phi^{\perp} \mathrm{w}\bigr)\bigr|^2 + \int\limits_{B(0,\delta/\varepsilon)} \bigl|\nabla \phi^{\perp}\bigr|^2 \Biggr\} \leqslant B\bigl(\phi^{\perp}, \phi^{\perp}\bigr), \tag{4.14}$$

where the orthogonality conditions

$$\operatorname{Re} \int\limits_{B(0,1/2)} \phi^{\perp} \bar{\mathrm{w}}_{x_j} = 0, \quad j = 1, 2,$$

are used. Now, since $B(\hat{\phi}^{\perp}, \hat{\phi}^{\perp}) = o(1)$, it follows that

$$\int\limits_{B(0,3R_0)} \bigl|\hat{\phi}^{\perp}\bigr|^2 = o(1)$$

and elliptic estimates yield $\hat{\phi}^{\perp} \to 0$ in C^1-sense in $B(0, 2R_0)$. Let us consider now $\hat{\phi}^0 = i\mathrm{w}\hat{\psi}^0$. Then

$$\Delta \hat{\psi}^0 + 2\frac{\nabla \mathrm{w}}{\mathrm{w}} \nabla \hat{\psi}^0 - 2i|\mathrm{w}|^2 \hat{\psi}_2^0 = H^0 \quad \text{in } B(0, \delta/\varepsilon),$$

$$\hat{\psi}^0 = 0 \quad \text{on } \partial B(0, \delta/\varepsilon).$$

This equation translates into the uncoupled system

$$\Delta \hat{\psi}_1^0 + \frac{2U'}{U} \frac{\partial \hat{\psi}_1^0}{\partial r} = H_1^0(r),$$

$$\Delta \hat{\psi}_2^0 + \frac{2U'}{U} \frac{\partial \hat{\psi}_2^0}{\partial r} - 2U^2 \hat{\psi}_2^0 = H_2^0(r)$$

for $0 < r < \frac{\delta}{\varepsilon}$. The first equation, plus the boundary condition has the unique solution

$$\hat{\psi}_1^0(r) = -\int\limits_r^{\delta/\varepsilon} \frac{ds}{sU^2(s)} \int\limits_0^s H_1^0(t) U^2(t) t \, dt.$$

Since

$$H_1^0(r) = \frac{o(1)}{|\log \varepsilon| r^{2+\sigma}} + o(\varepsilon^2), \quad r > 1,$$

$H_1^0(r) = o(1)\frac{1}{r}$, $r < 1$, it follows from the above formula that $\hat{\psi}_1^0(r) = o(1)$. On the other hand, a barrier shows that on $\hat{\psi}_1^0$ we have the estimate $\hat{\psi}_1^0(r) = o(1)r$. As a conclusion we finally derive

$$\int_{B(0,3R_0)} |\hat{\phi}^0|^2 + |\hat{\phi}^\perp|^2 = o(1),$$

and hence, from elliptic estimates, $\hat{\phi}_j \to 0$ in C^1-sense on $B(0, 2R_0)$. The final conclusion is that actually $\|\psi\|_* \to 0$. This is a contradiction with $\|\psi\|_* = 1$, and the result has been proven. □

We consider now the following linear problem.

$$\mathcal{L}^\varepsilon(\psi) = h + c_0\varepsilon^2 \chi_{\Omega_\varepsilon \setminus \bigcup_{j=1}^k B(\xi_j', \frac{\delta}{\varepsilon})} + \sum_{j,l} c_{jl} \frac{1}{iV_0} \mathrm{w}_{x_l}(y - \xi_j') \chi_{\{r_j < 1/2\}} \quad \text{in } \Omega_\varepsilon, \tag{4.15}$$

$$\frac{\partial \psi}{\partial \nu} = g \quad \text{on } \partial\Omega_\varepsilon, \tag{4.16}$$

$$\int_{\Omega_\varepsilon \setminus \bigcup_{j=1}^k B(\xi_j', \delta/\varepsilon)} \psi_1 = 0, \qquad \operatorname{Re} \int_{|z|<1} \bar{\phi}_j w_{x_l} = 0, \quad \forall j, l,$$

$$\phi_j(z) = i\mathrm{w}(z)\psi(\xi_j' + z). \tag{4.17}$$

Here we have called (with some abuse of notation) $\mathrm{w}(z) = \mathrm{w}^\pm(z)$ if $j \in I_\pm$. The following is the main result of this section.

Proposition 4.1. *There exists a constant $C > 0$, dependent on δ and Ω but independent of c_0, such that for all small ε the following holds*: *if $\|h\|_{**} + \|g\|_{***} < +\infty$ then there exists a unique solution $\psi = T_\varepsilon(h, g)$ to problem* (4.15)–(4.17). *Besides,*

$$\|T_\varepsilon(h, g)\|_* \leqslant C\left[|\log \varepsilon| \|h\|_{**} + \|g\|_{***}\right]. \tag{4.18}$$

Moreover, the constants c_{lj} admit the asymptotic expression

$$c_{lj} = -c_*^{-1} \operatorname{Re} \int_{\{|z|<\delta/\varepsilon\}} h_j \bar{\mathrm{w}}_{x_l} + O(\varepsilon \log \varepsilon)\|\psi\|_*, \tag{4.19}$$

where c_* is the constant in (4.22). *Here*

$$h_j(z) = \alpha_j(z)^{-1} h(\xi_j' + z).$$

Proof. Expressed in terms of $\phi = iV_0\psi$, the weak H^1 formulation of this problem can be written via Riesz's theorem in the form $\phi + K(\phi) = S$ where K is a linear, compact operator in the closed subspace of functions of $H^1(\Omega_\varepsilon)$ which satisfy the orthogonality conditions

$$\operatorname{Re} \int_{\{|z|<1/2\}} \phi_j \bar{\mathrm{w}}_{x_l} = 0 \quad \text{for all } l, j.$$

In fact, let us consider the space

$$H = \left\{ \phi \in H_0(\Omega_\varepsilon) \,\middle|\, \operatorname{Re} \int_{\{|z|<1/2\}} \phi_j \bar{\mathrm{w}}_{x_l} = 0 \text{ for all } l, j \right\}$$

endowed with the usual inner product $[\phi, \psi] = \int_{\Omega_\varepsilon} \nabla\phi\nabla\psi$. Problem (4.15)–(4.17) expressed in weak form is equivalent to that of finding a $\phi \in H$ such that

$$[\phi, \psi] = \big\langle \big(k(x)\phi - s\big), \psi \big\rangle \quad \forall \psi \in H.$$

With the aid of Riesz's representation theorem, this equation gets rewritten in H in the operational form

$$\phi + K(\phi) = S$$

with certain $S \in H$ which depends linearly in s and where K is a compact operator in H.

Fredholm alternative then yields the existence assertion, provided that the homogeneous equation only has the trivial solution. But this is a direct consequence from the estimate in Lemma 4.1 if we establish the a priori estimate (4.19).

In $|y - \xi_j'| \leqslant \frac{\delta}{\varepsilon}$, Eq. (4.15) is equivalent to

$$L_j^\varepsilon(\phi_j) = \tilde{h}_j + \sum_l c_{jl} \mathrm{w}_{x_l} \chi_{\{|z|<1/2\}}. \tag{4.20}$$

Here we have denoted $\tilde{h}_j(z) = i\mathrm{w}(z)h(\xi_j + z)$. Multiplying Eq. (4.20) against $\bar{\mathrm{w}}_{x_m}(y - \xi_j')$, integrating all over $B(0, \frac{\delta}{\varepsilon})$ and taking real parts one gets,

$$\operatorname{Re} \int_{B(0,\delta/\varepsilon)} L_j^\varepsilon(\phi_j) \bar{\mathrm{w}}_{x_m} = \operatorname{Re} \int_{B(0,\delta/\varepsilon)} \tilde{h}_j \bar{\mathrm{w}}_{x_m} + c_{jm} c_*, \tag{4.21}$$

where

$$c_* = \int_{B(0,1/2)} |\mathrm{w}_{x_m}|^2. \tag{4.22}$$

The desired result will follow from estimating the left-hand side of equality (4.21).

Integrating by parts, we write

$$\operatorname{Re}\int_{B(0,\delta/\varepsilon)} L_j^{\varepsilon}(\phi_j)\bar{\mathrm{w}}_{x_m} = \operatorname{Re}\left\{\int_{\partial B(0,\delta/\varepsilon)} \frac{\partial\phi_j}{\partial\nu}\bar{\mathrm{w}}_{x_m} - \int_{\partial B(0,\delta/\varepsilon)} \phi_j\frac{\partial}{\partial\nu}\bar{\mathrm{w}}_{x_m}\right\}$$
$$+\operatorname{Re}\int_{B(0,\delta/\varepsilon)} \bar{\phi}_j\big(L_j^{\varepsilon} - L^0\big)\mathrm{w}_{x_m}. \tag{4.23}$$

The boundary integrals can be estimated as

$$\left|\operatorname{Re}\left\{\int_{\partial B(0,\delta/\varepsilon)} \frac{\partial\phi_j}{\partial\nu}\bar{\mathrm{w}}_{x_m} - \int_{\partial B(0,\delta/\varepsilon)} \phi_j\frac{\partial}{\partial\nu}\bar{\mathrm{w}}_{x_m}\right\}\right| \leqslant C\varepsilon\|\psi\|_*.$$

The remaining term in (4.23) can be estimated in the following way:

$$\operatorname{Re}\int_{B(0,\delta/\varepsilon)} \big(L^{\varepsilon} - L^0\big)\mathrm{w}_{x_m}\bar{\phi}_j = \operatorname{Re}\int_{B(0,\delta/\varepsilon)} \big(\nabla\alpha_j\nabla\mathrm{w}_{x_m}$$
$$+\Delta\alpha_j\mathrm{w}_{x_m} + O\big(\varepsilon^2\big)\mathrm{w}_{x_m}\big)\bar{\phi}_j.$$

So we get

$$\left|\operatorname{Re}\int_{B(0,\delta/\varepsilon)} \big(L^{\varepsilon} - L^0\big)\mathrm{w}_{x_m}\bar{\phi}_j\right| \leqslant C\left|\int_1^{\delta/\varepsilon}\left(\frac{\varepsilon}{r^2} + \frac{\varepsilon^2}{r}\right)r\,dr\right|\|\phi\|_\infty$$
$$\leqslant C\varepsilon|\log\varepsilon|\|\psi\|_*.$$

Combining the above estimates we obtain the validity of (4.19). In particular, it readily follows that

$$|c_{jl}| \leqslant C\big[\|h\|_{**} + \|g\|_{***} + \varepsilon|\log\varepsilon|\|\psi\|_*\big]. \tag{4.24}$$

On the other hand, applying Lemma 4.1 one gets

$$\|\psi\|_* \leqslant C\big[|\log\varepsilon|\|h\|_{**} + |\log\varepsilon||c_{jl}| + \|g\|_{***}\big]. \tag{4.25}$$

Estimate (4.18) then follows combining (4.24) and (4.25). □

Remark 4.1. The result of Proposition 4.1 holds true, with no significant changes in the proof, for the Dirichlet problem

$$\mathcal{L}^{\varepsilon}(\psi) = h + \sum_{j,l} c_{jl}\frac{1}{iV_0}\mathrm{w}_{x_l}\big(y - \xi_j'\big)\chi_{\{r_j<1/2\}} \quad \text{in } \Omega_\varepsilon,$$

$$\psi = 0 \quad \text{on } \partial\Omega_\varepsilon,$$

$$\mathrm{Re} \int\limits_{|z|<1} \bar{\phi}_j w_{x_l} = 0, \quad \forall j, l,$$

where V_0 is defined from $w_{g\varepsilon}$ instead of $w_{\mathcal{N}\varepsilon}$. In fact, proofs go through the same way, without the need of introducing the parameter c_0 or the extra outer orthogonality condition.

5. The projected nonlinear problem

Our goal is to solve problem (3.5), (3.6) for a suitable small ψ. Rather than doing this directly, we consider first its projected version, for $\xi \in \Omega_\delta^k$,

$$\mathcal{L}^\varepsilon(\psi) = N(\psi) + R + \sum_{j,l} c_{jl} \frac{1}{i V_0} \mathrm{w}_{x_l}\bigl(x - \xi_j'\bigr) \chi_{\{r_j < 1/2\}} + c_0 \varepsilon^2 \chi_{\Omega_\varepsilon \setminus \bigcup_{j=1}^k B(\xi_j', \delta/\varepsilon)} \quad \text{in } \Omega_\varepsilon, \tag{5.1}$$

$$\frac{\partial \psi}{\partial \nu} = \frac{1}{i V_0} \frac{\partial V_0}{\partial \nu} \quad \text{on } \partial\Omega_\varepsilon, \tag{5.2}$$

$$\int\limits_{\Omega \setminus \bigcup_{j=1}^k B(\xi_j', \delta/\varepsilon)} \psi_1 = 0, \qquad \mathrm{Re} \int\limits_{|y|<1} \bar{\phi}_j \mathrm{w}_{x_l} = 0, \quad \forall j, l,$$

$$\phi_j(z) = i \mathrm{w}(z) \psi\bigl(\xi_j' + z\bigr). \tag{5.3}$$

We prove

Proposition 5.1. *There is a constant $C > 0$ depending only on δ and Ω such that for all points $\xi \in \Omega_\delta^k$ and ε small problem* (5.1)–(5.3) *possesses a unique solution ψ with*

$$\|\psi\|_* \leqslant C\varepsilon^{1-\sigma}.$$

Moreover, automatically one has that $c_0 = 0$.

Proof. As we computed in Lemma 2.1, we see that the boundary condition for ψ becomes in real and imaginary parts $\partial\psi_1/\partial\nu = 0$ and

$$\frac{\partial \psi_2}{\partial \nu} = S_2$$

with

$$S_2(y) = O\bigl(\varepsilon^3\bigr), \qquad \nabla S_2(y) = O\bigl(\varepsilon^4\bigr)$$

uniformly on $\partial\Omega_\varepsilon$. As for the error $R = R_1 + iR_2$, Lemma 2.1 yields

$$R_1 = O\left(\varepsilon\sum_{j=1}^{k}\frac{1}{r_j^3}\right), \qquad R_2 = O\left(\varepsilon\sum_{j=1}^{k}\frac{1}{r_j}\right)$$

if $r_j > 1$ for all j. Calling $\tilde{R}_j$ the error in ϕ_j-coordinates (see (3.24)) we also find

$$\|\tilde{R}_j\|_{C^{0,\gamma}(|z|<3)} = O(\varepsilon),$$

and then we conclude

$$\|R\|_{**} \leqslant C\varepsilon^{1-\sigma}.$$

Here and in what follows C denotes a generic constant independent of ε. We make the following claim: if $\|\psi\|_* \leqslant C\varepsilon^{\sigma}$ then $\|N(\psi)\|_{**} \leqslant C\varepsilon^{2-2\sigma}$. In fact, for $r_j > 2$ for all j, $N(\psi)$ reduces to

$$N(\psi)_1 = -2\nabla\psi_1\nabla\psi_2, \qquad N(\psi)_2 = |\nabla\psi_1|^2 - |\nabla\psi_2|^2 + |V_0|^2\big(e^{-2\psi_2} + 1 - 2\psi_2\big)$$

(see (3.9)). The definitions of the $*$-norm easily yields that in this region

$$\big|N(\psi)_1\big| \leqslant C\varepsilon^{2-2\sigma}\frac{1}{r_j^{2+\sigma}}, \qquad \big|N(\psi)_2\big| \leqslant C\varepsilon^{2-2\sigma}\frac{1}{r_j^2}.$$

On the other hand, calling $\tilde{N}_j(\phi_j)$ the operator in the ϕ_j-variable, as defined in (3.25) we see that

$$\tilde{N}_j(\phi_j) = A_1(z, \phi_j, \nabla\phi_j) + A_2\big(z, \phi_j, \nabla\phi_j, D^2\phi_j\big),$$

where A_i are smooth functions of their arguments, with A_2 supported only for $|z| < 2$, and with

$$\big|A_1(z, p, q)\big| \leqslant C\big[|p|^2 + |q|^2\big], \qquad \big|A_2(z, p, q, r)\big| \leqslant C\big[|p|^2 + |q|^2 + |r|^2\big]$$

near $(p, q, r) = (0, 0, 0)$. By assumption we have

$$\|\phi_j\|_{C^{2,\gamma}(|z|<2)} + \|\phi_j\|_{C^{1,\gamma}(|z|<3)} \leqslant C\varepsilon^{1-\sigma},$$

from where it is straightforward to check $\|\tilde{N}_j(\phi_j)\|_{C^{0,\gamma}(|z|<2)} \leqslant C\varepsilon^{2-2\sigma}$, and the claim is proven.

On the other hand, it is also true that if $\|\psi^\ell\|_* \leqslant C\varepsilon^{1-\sigma}$ for $\ell = 1, 2$ then

$$\big\|N\big(\psi^1\big) - N\big(\psi^2\big)\big\|_{**} \leqslant C\varepsilon^{\frac{1-\sigma}{2}}\big\|\psi^1 - \psi^2\big\|_*.$$

Problem (5.1)–(5.3) is equivalent to the fixed point problem

$$\psi = T_\varepsilon\big(N(\psi) + R, iS_2\big),$$

where T_ε is the linear operator introduced in Proposition 4.1. Since $\|T_\varepsilon\| = O(\log \varepsilon)$, the above estimates yield a unique solution with size $\|\psi\|_* \leqslant C\varepsilon^{1-\sigma}$. Hence we have proven the existence of a unique solution in this range for problem (5.1)–(5.3).

Let us observe now the following. If ψ satisfies (5.1)–(5.3) then v given by (3.1) satisfies

$$\Delta v + \left(1 - |v|^2\right)v = c_0\varepsilon^2 i v \chi_{\mathrm{out}} + \sum_{j,l} c_{jl}\alpha_j^{-1} \mathrm{w}_{x_l}\left(y - \xi_j'\right)\chi_{\{r_j<1/2\}}.$$

Here $\chi_{\mathrm{out}} = \chi_{\Omega_\varepsilon \setminus \bigcup_{j=1}^k B(\xi_j', \delta/\varepsilon)}$. Moreover, if $r_j < \frac{1}{2}$ then $v(\xi_j' + z) = \alpha_j[\mathrm{w}(z) + \phi_j]$. Hence, multiplying the above equation by $\bar{v}$, using the orthogonality conditions assumed and integrating by parts we get

$$-\int_{\Omega_\varepsilon} |\nabla v|^2 + \int_{\Omega_\varepsilon} \left(1 - |v|^2\right)|v|^2 = i c_0 \varepsilon^2 \int_{\Omega_\varepsilon} |v|^2 \chi_{\mathrm{out}}.$$

The conclusion is that automatically $c_0 = 0$. The proof is concluded. $\square$

The function $\psi(\xi)$ defined in the above proposition turns out to be continuously differentiable as we argue next.

Emphasizing the dependence on ξ' in the fixed point characterization of ψ, we write

$$\psi = T_\varepsilon(\xi')\left(R(\xi') + N(\psi, \xi')\right) \equiv M(\psi, \xi).$$

Somewhat lengthy but straightforward verification yields differentiability of the operator in the right-hand side of the above equation in the variables (ψ, ξ') for the norms considered. In particular, the fixed point characterization renders continuity of $\psi(\xi')$ in the *-norm.

Formally, the partial derivative $\partial_{\xi_{kl}'}\psi$ satisfies

$$\partial_{\xi_{kl}'}\psi = \left(\partial_{\xi_{kl}'} T_\varepsilon(\xi')\right)\left(N(\psi, \xi') + R(\xi')\right) + T_\varepsilon(\xi')\left(\partial_{\xi_{kl}'}\left[N(\psi, \xi') + R(\xi')\right]\right),$$

equation that takes the form

$$\left(I - T_\varepsilon(\xi')\left((D_\psi N)(\psi, \xi)\right)\right)[\partial_{\xi_{kl}'}\psi] = H(\xi') \tag{5.4}$$

for a continuous function $H(\xi')$. The estimate

$$\left\| D_\psi N(\psi, \xi)[\zeta]\right\|_{**} \leqslant C\varepsilon^{\sigma'}\|\zeta\|_*$$

for some $0 < \sigma' < 1$ is found from direct computation. Since we also have $\|T_\varepsilon\| \leqslant C|\log \varepsilon|$, it follows that the linear operator on the left-hand side of (5.4) is invertible for all small ε and hence one can solve for $\partial_{\xi_{kl}'}\psi$. But this condition is precisely that making the implicit function theorem applicable, so that ψ is indeed of class C^1 as a function of ξ'.

In order to construct a solution to (3.5) corresponding in original space variable to that predicted by Theorem 1.1, what we need to do is to find $\xi \in \mathcal{D}$ in such a way that $c_{jl} = 0$ for all j, l in (5.1)–(5.3). As we will see, this problem is equivalent to a variational problem neighboring that of finding critical points of the renormalized energy $W_{\mathcal{N}}$. We carry out this conclusion in the next two sections.

6. Role and expressions of renormalized energies

In this section we will compute expansions for the quantities $J_\varepsilon(w_{\mathcal{N}\varepsilon})$ and $J_\varepsilon(w_{g\varepsilon})$. We shall carry out the computation for the Neumann problem. The Dirichlet case is similar, and it is essentially contained in [3].

Let us observe that $\varphi_{\mathcal{N}}$ given by (1.17), (1.18) can be represented as

$$\varphi_{\mathcal{N}}(x,\xi,\mathbb{d}) = \sum_{j=1}^{k} d_j \varphi_j(x),$$

where $\varphi_j(x)$ solves

$$\Delta\varphi_j = 0 \quad \text{in } \Omega,$$

$$\frac{\partial\varphi_j}{\partial\nu} = -\frac{(x-\xi_j)^\perp\cdot\nu}{|x-\xi_j|^2} \quad \text{on } \partial\Omega, \qquad \int_\Omega \varphi_j = 0.$$

Let Γ_0 be the outer component of $\partial\Omega$, and let us denote by Γ_l, $l=1,\dots,n$, its inner components, if any. Let us observe that $\int_{\Gamma_l} \partial\varphi_j/\partial\nu = 0$, for all $l=1,\dots,n$. These relations imply the existence of a *harmonic conjugate* for φ_j, namely a harmonic function $\varphi_j^\perp$ that satisfies

$$\partial_{x_2}\varphi_j^\perp = \partial_{x_1}\varphi_j, \qquad \partial_{x_1}\varphi_j^\perp = -\partial_{x_2}\varphi_j.$$

Observe that this harmonic conjugate thus satisfies on each Γ_l the relation

$$\partial_\tau\left(\varphi_j^\perp(x) + \log\frac{1}{|x-\xi_j|}\right) = 0$$

and hence there is a number $c_l(\xi_j)$ such that

$$\varphi_j^\perp(x) + \log\frac{1}{|x-\xi_j|} = c_l(\xi_j) \quad \text{on } \Gamma_l.$$

Since harmonic conjugate is defined up to an additive constant, we impose $c_0(\xi_j)\equiv 0$. Now, let $\phi_l(x)$ be the solution of the boundary value problem

$$\Delta\phi_l = 0 \quad \text{in } \Omega,$$

$$\phi_l = \delta_{lj} \quad \text{on } \Gamma_j,\ \forall j \geqslant 0.$$

Let $G_0(x,\xi)$ denote the Green's function for the problem

$$-\Delta G_0 = 2\pi\delta(x-\xi_j) \quad \text{in } \Omega,$$

$$G_0(x,\xi) = 0 \quad \text{on } \partial\Omega,$$

and $H_0(x,\xi)$ its regular part,

$$H_0(x,\xi) = \log\frac{1}{|x-\xi|} - G_0(x,\xi).$$

Then we can represent

$$\varphi_j^{\perp}(x) + \log\frac{1}{|x-\xi_j|} = \sum_{l=1}^{n} c_l(\xi_j)\phi_l(x) + G_0(x,\xi_j).$$

The existence of a harmonic conjugate for $\varphi_j^{\perp}$ implies that the mean value of its normal derivative must be zero on each component Γ_l, $l \geqslant 1$. This yields the relations

$$c_l(\xi_j)\int_{\Gamma_l} \partial_\nu \phi_l(x) + \int_{\Gamma_l} \partial_\nu G_0(x,\xi_j) = 0.$$

We observe that the following identity holds:

$$2\pi\phi_l(\xi) = \int_{\Gamma_l} \partial_\nu G_0(x,\xi_j).$$

Then if we set

$$\gamma_l \equiv 2\pi\left(\int_{\Gamma_l} \partial_\nu \phi_l\right)^{-1},$$

we get $c_l(\xi_j) = \gamma_l\phi_l(\xi)$. Let us denote

$$G(x,\xi) = \sum_{l=1}^{n} \gamma_l\phi_l(\xi)\phi_l(x) + G_0(x,\xi), \tag{6.1}$$

where the sum is understood to be zero if the domain is simply connected. Consistently we denote

$$H(x,\xi) = -\sum_{l=1}^{n} \gamma_l\phi_l(\xi)\phi_l(x) + H_0(x,\xi). \tag{6.2}$$

Let us write $W(x-\xi_j/\varepsilon) = U(r_j/\varepsilon)e^{i\theta_j}$ where U is the solution of (1.11) and (r_j,θ_j) are polar coordinates relative to ξ_j. We decompose

$$w_{\mathcal{N}\varepsilon}(x) = U_\varepsilon(x)e^{i\Phi(x)},$$

where $\Phi = \sum_{j=1}^{k} d_j(\varphi_j + \theta_j)$ and $U_\varepsilon(x) = \prod_{j=1}^{k} U(r_j/\varepsilon)$. Then we have

$$\int_\Omega |\nabla w_{\mathcal{N}\varepsilon}|^2 = \int_\Omega |\nabla U_\varepsilon|^2 + \int_\Omega |\nabla\Phi|^2 U_\varepsilon^2 = I_1 + I_2.$$

Now,

$$I_1 = k \int_{\mathbb{R}^2} |\nabla U|^2 + O(\varepsilon^2)$$

and, denoting $G_j(x) = G(x, \xi_j)$, we have

$$I_2 = \int_\Omega \left| \sum_{j=1}^k d_j \nabla G_j \right|^2 U_\varepsilon^2 = \sum_{i \neq j} \int_\Omega d_i d_j \nabla G_i \nabla G_j U_\varepsilon^2 + \sum_{j=1}^k \int_\Omega |\nabla G_j|^2 U_\varepsilon^2 = I_{21} + I_{22}.$$

We easily compute that

$$\int_\Omega \nabla G_i \nabla G_l (1 - U_\varepsilon^2) = O(\varepsilon),$$

while, integrating by parts,

$$\int_\Omega \nabla G_i \nabla G_j = 2\pi G(\xi_i, \xi_j),$$

since $\int_{\partial\Omega} \partial_\nu G_i G_j = 0$. In fact, we have $\int_{\Gamma_l} \partial_\nu G_i = 0$ for all $l \neq 0$ while G_j is constant on Γ_l. Besides, $G_j = 0$ on Γ_0. It follows that

$$I_{21} = 2\pi \sum_{i \neq j} d_i d_j G(\xi_i, \xi_j) + O(\varepsilon).$$

Now, consider a small number $\delta \gg \varepsilon$, to be fixed later. We have

$$\int_\Omega |\nabla G_j|^2 U_\varepsilon^2 = \int_{B_\delta(\xi_j)} |\nabla G_j|^2 U_\varepsilon^2 + \int_{\Omega \setminus B_\delta(\xi_j)} |\nabla G_j|^2 + O(\delta^{-2}\varepsilon^2).$$

Now,

$$\begin{aligned}
\int_{\Omega \setminus B_\delta(\xi_j)} |\nabla G_j|^2 &= \int_{\partial B_\delta(\xi_j)} G_j \partial_\nu G_j \\
&= \int_{\partial B_\delta(\xi_j)} \left(-H(x, \xi_j) + \log \frac{1}{\delta}\right)\left(\frac{1}{\delta} - \partial_\nu H(x, \xi_j)\right) \\
&= -2\pi H(\xi_j, \xi_j) + 2\pi \log \frac{1}{\delta} + O(\delta^2).
\end{aligned}$$

On the other hand, we have that

$$\int_{B_\delta(\xi_j)} |\nabla G_j|^2 U_\varepsilon^2 = O(\delta^2) + 2\pi \int_0^\delta U^2(r/\varepsilon)\,\frac{dr}{r} = O(\delta^2) + c_0 + 2\pi \log\frac{\delta}{\varepsilon},$$

where c_0 is an universal constant. Let us make now the choice $\delta = \varepsilon^{1/2}$. Combining the above expansions we then get

$$I_{22} = -\sum_{j=1}^{k} 2\pi H(\xi_j, \xi_j) + 2k\pi \log\frac{1}{\varepsilon} + c_1 + O(\varepsilon)$$

and in total

$$J_\varepsilon\big(w_{\mathcal{N}\varepsilon}(\cdot, \xi, \mathrm{d})\big) = k\pi \log\frac{1}{\varepsilon} + W_{\mathcal{N}}(\xi, \mathrm{d}) + c_2 + O(\varepsilon), \tag{6.3}$$

where

$$W_{\mathcal{N}}(\xi, \mathrm{d}) = \pi \sum_{i\neq j} d_i d_j G(\xi_i, \xi_j) - \sum_{j=1}^{k} \pi H(\xi_j, \xi_j) \tag{6.4}$$

and c_1, c_2 are absolute constants which depend on the number k of points. Actually examining the expressions in the expansions above we also see that it holds in the C^1-sense, namely

$$\nabla_\xi J_\varepsilon\big(w_{\mathcal{N}\varepsilon}(\cdot, \xi, \mathrm{d})\big) = \nabla_\xi W_{\mathcal{N}}(\xi, \mathrm{d}) + O(\varepsilon). \tag{6.5}$$

Expression (6.4) for the renormalized energy in the Neumann problem was derived in [18] as a tool to analyze formally dynamics of vortices, in the simply connected case, $G = G_0$. Also in the simply connected situation, it appears in the analysis in [33]. Estimate (6.3) is also pushed to the C^1 and C^2 orders in [31,33]. Examining the above proof, we observe that $W_{\mathcal{N}}$ corresponds precisely to expression (1.19).

As we have mentioned, the corresponding computation for the Dirichlet problem is basically contained in [3]. In such a case, the following expansion is found:

$$J_\varepsilon\big(w_{g\varepsilon}(\cdot, \xi, \mathrm{d})\big) = k\pi \log\frac{1}{\varepsilon} + W_g(\xi, \mathrm{d}) + c_3 + O(\varepsilon), \tag{6.6}$$

where now W_g is given by formula (1.8), which corresponds to the following explicit description. Let Φ be the unique solution of the problem

$$\Delta\Phi = 2\pi \sum_{j=1}^{k} d_j \delta(x - \xi_j) \quad \text{in } \Omega, \tag{6.7}$$

$$\frac{\partial\Phi}{\partial\nu} = g \times g_\tau \quad \text{on } \partial\Omega, \qquad \int_\Omega \Phi = 0, \tag{6.8}$$

and set

$$R(x) = \Phi(x) - \sum_{j=1}^{k} d_j \log|x - \xi_j|. \tag{6.9}$$

Then

$$W_g(\xi, \mathrm{d}) = -\pi \sum_{l \neq j} d_l d_j \log|\xi_i - \xi_j| - \pi \sum_{j=1}^{k} d_j R(\xi_j) + \frac{1}{2} \int_{\partial\Omega} \Phi(g \times g_\tau). \tag{6.10}$$

7. Variational reduction

Let us consider the equations $c_{jl}(\xi) = 0$ in (5.1)–(5.3) for the solution $\psi = \psi(\xi)$ predicted by Proposition 5.1. We denote by $v(\xi)$ the ansatz (3.1) for this ψ and consider the functional

$$P_\varepsilon(\xi) = J_\varepsilon\big(v(\xi)\big).$$

Next proposition states that the above system of equations corresponds to finding critical points of P_ε. Moreover, asymptotics of P_ε in terms of renormalized energy hold in C^1-sense.

Proposition 7.1.

(a) *If $\nabla_\xi P(\xi) = 0$ then $c_{jl}(\xi) = 0$ for all j, l.*
(b) *We have the validity of the expansion*

$$\nabla_\xi P_\varepsilon(\xi) = \nabla_\xi W_{\mathcal{N}}(\xi, \mathrm{d}) + O\big(\varepsilon^{1-\sigma}\big), \tag{7.1}$$

uniformly on $\xi \in \Omega_\delta^k$.

Proof. We write $\xi = (\xi_1, \ldots, \xi_k)$. We also denote $\xi_j = (\xi_{j1}, \xi_{j2})$ and $\xi'_j = (\xi'_{j1}, \xi'_{j2})$. We have

$$\begin{aligned} -\partial_{\xi'_{j_0 i_0}} P_\varepsilon(\xi) &= -J'_\varepsilon\big(v(\xi)\big)[v_{\xi'_{j_0 l_0}}] = \operatorname{Re} \int_{\Omega_\varepsilon} \big[\Delta v + \big(1 - |v|^2\big)v\big] \bar{v}_{\xi'_{j_0 l_0}} \\ &= \sum_{l,j} c_{jl} \operatorname{Re} \int_{|z|<1} \frac{\alpha_j}{|\alpha_j|^2} \mathrm{w}_{x_l}(z) \bar{v}_{\xi'_{j_0 l_0}}\big(\xi'_j + z\big). \end{aligned}$$

Now, near ξ'_j we have

$$\begin{aligned} v_{\xi_{j_0 l_0}}(y) &= \partial_{\xi'_{j_0 l_0}} \big[\alpha_j\big(y - \xi'_j, \xi\big)\big(\mathrm{w}\big(y - \xi'_j\big) + \phi_j\big(y - \xi'_j, \xi\big)\big)\big] \\ &= (\partial_{\xi'_{j_0 i_0}} \alpha_j)(\mathrm{w} + \phi_j) + \alpha_j(\partial_{\xi'_{j_0 l_0}} \phi_j) - \delta_{jj_0} \partial_{z_{l_0}}(\alpha_j \mathrm{w} + \alpha_j \phi_j). \end{aligned}$$

We observe that both $\partial_z\alpha_j$ and $\partial_{\xi'}\alpha_j$ are functions of size $O(\varepsilon)$ in the region $|z|<\frac{1}{2}$. As for ϕ_j and $\partial_z\phi_j$, they are of order $O(\varepsilon^{1-\sigma})$. On the other hand, we know that

$$\operatorname{Re}\int\limits_{B(0,1/2)} \phi_j(z,\xi)\bar{\mathrm{w}}_z(z)\,dz = 0 \quad \text{and hence} \quad \operatorname{Re}\int\limits_{B(0,1/2)} \partial_{\xi'_{j_0l_0}}\phi_j(z,\xi)\bar{\mathrm{w}}_{x_l}(z)\,dz = 0.$$

It holds as well

$$\operatorname{Re}\int\limits_{B(0,1/2)} \bar{\mathrm{w}}_{z_{l_0}}\mathrm{w}_{z_l} = c^*\delta_{l_0l},$$

where $c^* = \int_{B(0,1/2)} |\mathrm{w}_{z_1}|^2\,dz$. Combining the above facts we then find

$$-\partial_{\xi'_{j_0l_0}} P_\varepsilon(\xi) = \sum_{l,j} c_{jl}\big[c^*\delta_{j_0l_0} + O\big(\varepsilon^{1-\sigma}\big)\big],$$

from where part (a) follows.

According to Proposition 4.1, we have the validity of the estimate

$$c^*c_{ij} = \operatorname{Re}\int\limits_{\{|z|<\delta/\varepsilon\}} \big(\tilde{N}_j(\phi_j) + \tilde{R}_j\big)\bar{\mathrm{w}}_{x_i} + O\big(\varepsilon^{2-2\sigma}\big),$$

cf. (3.24) and (3.25). On the other hand,

$$\begin{aligned}\operatorname{Re}\int\limits_{\{|z|<\delta/\varepsilon\}} \big(\tilde{N}_j(\phi_j) + \tilde{R}_j\big)\bar{\mathrm{w}}_{x_i} &= \int\limits_{\Omega_\varepsilon} \big[\Delta V_0 + \big(1-|V_0|^2\big)\big](V_0)_{\xi'_{ji}} + O\big(\varepsilon^{2-2\sigma}\big)\\ &= \partial_{\xi'_{ji}} J_\varepsilon(V_0) + O\big(\varepsilon^{2-2\sigma}\big).\end{aligned}$$

But, according to expansion (6.5), we see that

$$\partial_{\xi'_{ji}} J_\varepsilon(V_0) = \varepsilon\partial_{\xi_{ji}} W_{\mathcal{N}}(\xi,\mathrm{d}) + O\big(\varepsilon^{2-2\sigma}\big).$$

Combining the above estimates we find

$$\nabla_\xi P_\varepsilon(\xi) = \nabla_\xi W_{\mathcal{N}}(\xi,\mathrm{d}) + O\big(\varepsilon^{1-\sigma}\big)$$

and the proof is complete. □

8. Proof of main results

8.1. Proof of Theorem 1.1

Let us first consider the case of the Neumann problem. Let $\psi = \psi(\xi)$ be the solution of problem (5.1)–(5.3) predicted by Proposition 5.1. Then the function $v = v(\xi)$ given by (3.1) is a solution to (2.2) if we adjust the points ξ to that $c_{jl}(\xi)=0$ in (5.1)–(5.3). Proposition 7.1(a), says that this is equivalent to find a critical point for $P(\xi)=J_\varepsilon(v(\xi))$. Again Proposition 7.1(b), gives the validity of expansion (7.1), namely

$$\nabla_\xi P(\xi)=\nabla_\xi W_{\mathcal{N}}(\xi,\mathrm{d})+O\big(\varepsilon^{1-\sigma}\big)$$

uniformly on Ω_δ^k. Now, by assumption, the function $W_{\mathcal{N}}(\cdot,\mathrm{d})$ exhibits a non-trivial critical points situation in Ω_δ^k. By definition, then also $P(\xi)$ has a critical point the same region. This gives (1.22). The fact that (1.21) holds true follows by construction.

For the Dirichlet problem the proof is basically identical, taking into account Remark 4.1 for the associated linear problem.

8.2. *Proof of Theorem 1.2*

For part (a), we consider the choice $k=1$, $\mathrm{d}=(1)$. According to Theorem 1.1, it suffices to establish the presence of a set $\mathcal{D}\subset\Omega$ where $W_{\mathcal{N}}(\xi,\mathrm{d})$ has a non-trivial critical point situation. We observe that in this case $W_{\mathcal{N}}$ reduces just to

$$W_{\mathcal{N}}(\xi,\mathrm{d})=-\pi H_0(\xi,\xi)+\pi\sum_{l=1}^{n}\gamma_l\phi_l(\xi)^2,$$

where the second sum appears only if the domain is not simply connected. A standard fact on Robin's function $H_0(\xi,\xi)$ is that it approaches $+\infty$ as ξ gets close to the boundary $\partial\Omega$. The other term, in the above expression remains instead bounded. Thus if we choose

$$\mathcal{D}=\big\{\xi\in\Omega\mid \operatorname{dist}(\xi,\partial\Omega)>\delta\big\}$$

with δ sufficiently small, we obtain that

$$\sup_{\xi\in\mathcal{D}}W_{\mathcal{N}}(\xi,\mathrm{d})>\sup_{\xi\in\partial\mathcal{D}}W_{\mathcal{N}}(\xi,\mathrm{d}),$$

and a maximum situation for $W_{\mathcal{N}}$ is present in $\mathcal{D}$, which certainly remains for any small C^1-perturbation, and part (a) is proven.

For part (b) the argument is similar. Now let us take $k=2$ and $\mathrm{d}=(1,-1)$. Then $W_{\mathcal{N}}$ now becomes

$$W_{\mathcal{N}}(\xi,\mathrm{d})=\pi\left[-G(\xi_1,\xi_2)-\sum_{i=1}^{2}H(\xi_i,\xi_i)\right].$$

A maximum situation is now present in the region

$$\mathcal{D}=\big\{\xi\in\Omega^2\mid \operatorname{dist}\big(\xi,\partial\Omega^2\big)>\delta,\ |\xi_1-\xi_2|>\delta\big\}$$

if δ is chosen small enough. Moreover, if δ is very small, it is easily argued that $\mathcal{D}$ is topologically equivalent to the region $\Omega\times(\Omega\setminus\{P\})$, with P a point in Ω. The Ljusternik–Schnirelmann category of this region is at least two. Using that the boundary values are close to $-\infty$, together with the maximum value we have a second critical value,

$$\sup_{\mathrm{Cat}(A)\geqslant 2} \inf_A W_{\mathcal{N}}, \quad A \subset \mathcal{D}.$$

This second critical value does persist under small C^1 perturbations. If these two values are distinct, then two distinct critical points (modulo permutation of coordinates) are present. If for the perturbation they turn out to be equal, then standard theory gives that infinitely many critical points exist. This concludes the proof.

Now, for part (c), we consider the choice of $k = m \geqslant 2$ and $\mathtt{d} = \mathtt{1} = (1, \ldots, 1)$. According to the result of Theorem 1.1, it is sufficient to establish that $W_{\mathcal{N}}(\xi, \mathtt{1})$ has a non-trivial critical value in some open set $\mathcal{D}$, compactly contained in Ω^k. Our choice of $\mathcal{D}$ is just given by

$$\mathcal{D}_\delta = \bigl\{\xi \in \Omega^m \mid \mathrm{dist}\bigl(\xi, \partial\Omega^m\bigr) > \delta\bigr\},$$

where δ is a small positive number yet to be chosen. We observe that with no ambiguity, we may set $W_{\mathcal{N}}(\xi) = +\infty$ if $\xi_i = \xi_j$ for some $i \neq j$.

Let Ω_1 be a bounded non-empty component of $\mathbb{R}^2 \setminus \bar{\Omega}$, and consider a closed, smooth Jordan curve γ contained in Ω which encloses Ω_1. Let S to be the image of γ, and $B = S \times \cdots \times S = S^k$.

Then define

$$\mathcal{C} = \sup_{\Phi \in \Gamma} \inf_{z \in B} W_{\mathcal{N}}\bigl(\Phi(z), \mathtt{1}\bigr), \tag{8.1}$$

where $\Phi \in \Gamma$ if and only if $\Phi(z) = \Psi(1, z)$ with $\Psi : [0,1] \times B \to \mathcal{D}$ continuous and $\Psi(0, z) = z$.

We want to prove that this number defines a critical value for $W_{\mathcal{N}}$ in $\mathcal{D}$, and besides, for any small C^1-perturbation of this function. This is a consequence of the following two intermediate facts.

Claim 1. *There exists $K > 0$, independent of the small number δ used to define $\mathcal{D}_\delta$ such that $\mathcal{C} \leqslant K$.*

Claim 2. *Given $K > 0$, there exists $\delta > 0$ such that if $(\xi_1, \ldots, \xi_m) \in \partial\mathcal{D}_\delta$, and $|W_{\mathcal{N}}(\xi_1, \ldots, \xi_m, \mathtt{1})| \leqslant K$, then there is a vector τ, tangent to $\partial\mathcal{D}_\delta$ such that*

$$\nabla W_{\mathcal{N}}(\xi_1, \ldots, \xi_m, \mathtt{1}) \cdot \tau \neq 0.$$

From these facts the quantity $\mathcal{C}$ is finite, and from a standard deformation argument, it must define a critical value in $\mathcal{D}_\delta$. These conditions do survive for any small C^1-perturbation of $W_{\mathcal{N}}$ and hence Theorem 1.1 applies to yield the desired result.

To establish Claim 1, we need to prove the existence of $K > 0$ independent of small δ such that if $\Phi \in \Gamma$, then there exists $\bar{z} \in B$ with

$$W_{\mathcal{N}}\bigl(\Phi(\bar{z}), \mathtt{1}\bigr) \leqslant K. \tag{8.2}$$

Let us assume that $0 \in \Omega_1$ and write

$$\Phi(z) = \big(\Phi_1(z), \ldots, \Phi_m(z)\big).$$

Identifying the components of the above m-tuple with complex numbers, we shall establish the existence of $\bar{z} \in B$ such that

$$\frac{\Phi_j(\bar{z})}{|\Phi_j(\bar{z})|} = e^{\frac{2j\pi i}{m}} \quad \text{for all } j = 1, \ldots, m. \tag{8.3}$$

Clearly in such a situation, there is a number $\mu > 0$ depending only on m and Ω such that

$$\big|\Phi_j(\bar{z}) - \Phi_l(\bar{z})\big| \geqslant \mu.$$

This, and the definition of $W_{\mathcal{N}}$ clearly yields the validity of estimate (8.2) for a number K only dependent of Ω. To prove (8.3), one builds, based on the coordinates (8.3), a map of the torus T^m, which can be extended naturally to a map of a solid torus embedded in $\mathbb{R}^{m+1}$ which is homotopic to the identity. By a degree argument this map turns out to be onto which inherits in particular the existence of $\bar{z}$ as in (8.3).

As for Claim 2, let us assume the opposite, namely the existence of a sequence $\delta \to 0$ and of points $\xi = \xi^\delta$ for which $\xi \in \partial\mathcal{D}$ and such that

$$\nabla_{\xi_i} W_{\mathcal{N}}(\xi_1, \ldots, \xi_m, 1) = 0 \quad \text{if } \xi_i \in \Omega_\delta, \tag{8.4}$$

and

$$\nabla_{\xi_i} W_{\mathcal{N}}(\xi_1, \ldots, \xi_m, 1) \cdot \tau_i = 0 \quad \text{if } \xi_i \in \partial\Omega_\delta, \tag{8.5}$$

for any vector τ_i tangent to $\partial\Omega_\delta$ at ξ_i, where $\Omega_\delta = \{x \in \Omega\colon \operatorname{dist}(x, \partial\Omega) > \delta\}$.

From the assumption it follows that there is a point $\xi_l \in \partial\Omega_\delta$, such that $H(\xi_l, \xi_l) \to \infty$ as $\delta \to 0$. Since the value of $W_{\mathcal{N}}$ remains uniformly bounded, necessarily we must have that at least two points ξ_i and ξ_j that are becoming close. Let $\delta_n = \frac{1}{n}$, $\xi^n = (\xi_1^n, \ldots, \xi_m^n) \in \Omega_{\delta_n}$ be a sequence of points such that (8.4), (8.5) hold, and

$$\rho_n = \inf_{i \neq j} \big|\xi_j^n - \xi_i^n\big| \to 0, \quad \text{as } n \to \infty.$$

Without loss of generality, we can assume that $\rho_n = |\xi_1^n - \xi_2^n|$. We define

$$x_j^n = \frac{\xi_1^n - \xi_j^n}{\rho_n}. \tag{8.6}$$

Clearly there exists k, $2 \leqslant k \leqslant m$, such that

$$\lim_{n\to\infty} \big|x_j^n\big| < \infty, \quad j = 1, \ldots, k \quad \text{and} \quad \lim_{n\to\infty} \big|x_j^n\big| = \infty, \quad j > k.$$

For $j \leqslant k$ we set

$$\tilde{x}_j = \lim_{n\to\infty} x_j^n.$$

We consider two cases:

(1) either

$$\frac{\operatorname{dist}(\xi_1^n, \partial\Omega_{\delta_n})}{\rho_n} \to \infty;$$

(2) or there exists $c_0 < \infty$ such that for almost all n we have

$$\frac{\operatorname{dist}(\xi_1^n, \partial\Omega_{\delta_n})}{\rho_n} < c_0.$$

Case (1). It is easy to see that in this case we actually have

$$\frac{\operatorname{dist}(\xi_j^n, \partial\Omega_{\delta_n})}{\rho_n} \to \infty, \quad j = 1, \dots, k.$$

Furthermore, points $\xi_1^n, \dots, \xi_k^n$ are all interior to Ω_{δ_n} hence (8.4) is satisfied for all partial derivatives $\partial_{\xi_{lj}}$, $j \leqslant k$. Define

$$\tilde{W}_{\mathcal{N}}(x_1, \dots, x_m) = W_{\mathcal{N}}(\xi_1 + \rho_n x_1, \dots, \xi_1 + \rho_n x_m, \mathbb{1}).$$

We have for all $l = 1, 2$, $j = 1, \dots, k$,

$$\partial_{x_{lj}} \tilde{W}_{\mathcal{N}}(x) = \rho_n \partial_{\xi_{lj}} W_{\mathcal{N}}\big(\xi_1^n + x\rho_n, \mathbb{1}\big).$$

Then at $\tilde{x} = (\tilde{x}_1, \dots, \tilde{x}_k, 0, \dots, 0)$ we have

$$\partial_{x_{lj}} \tilde{W}_{\mathcal{N}}(\tilde{x}) = 0.$$

On the other hand, using the fact that

$$\big|\nabla_x H(x, y)\big| + \big|\nabla_y H(x, y)\big| \leqslant C_1 \min\left\{\frac{1}{|x - y|}, \frac{1}{\operatorname{dist}(y, \partial\Omega)}\right\} + C_2 \tag{8.7}$$

and letting $\rho_n \to 0$, we get

$$\lim_{n\to\infty} \rho_n \partial_{\xi_{lj}} W_{\mathcal{N}}\big(\xi_1^n + x\rho_n\big) = -4 \sum_{i\neq j,\, i\leqslant k} \partial_{x_{lj}} \log \frac{1}{|\tilde{x}_j - \tilde{x}_i|} = 0.$$

This last equality is true for any $j \leqslant k$, $l = 1, 2$. On the other hand, consider the function

$$\Psi_k(x_1, \dots, x_k) = -4 \sum_{i\neq j} \log|x_i - x_j|$$

defined for $x_j \in \mathcal{H} = \{(x^1, x^2)\colon\ x^1 \geqslant 0\}$. Denote I_+ the set of indices i for which $x_i^1 > 0$ and I_0 that for which $x_i^1 = 0$. Then explicit computations show that, either

$$\nabla_{x_i} \Psi_k(x_1, \dots, x_k) \neq 0, \quad \text{for some } i \in I_+, \tag{8.8}$$

or

$$\frac{\partial}{\partial x_{i2}}\Psi_k(x_1,\dots,x_k)\neq 0,\quad \text{for some } i\in I_0. \tag{8.9}$$

This fact proves impossibility of the case (1) above.

It remains to consider:

Case (2). In this case there exists a constant C such that

$$\frac{\operatorname{dist}(\xi_j^n,\partial\Omega_{\delta_n})}{\rho_n}\leqslant C,\quad j=1,\dots,k.$$

If there points ξ_j^n are all interior to Ω_{δ_n} then after scaling with ρ_n we argue as in case (1) and we reach a contradiction with the fact that the function $\bar\varphi_k$ given by

$$\bar\varphi_k(x_1,\dots,x_k)=4\sum_{i=1}^k\log\frac{1}{|x_i-\bar x_i|}+4\sum_{i\neq j}\log\frac{|x_i-x_j|}{|x_i-\bar x_j|}$$

has the property that

$$\nabla\bar\varphi_k(x_1,\dots,x_k)\neq 0$$

for any k distinct points $x_i\in\operatorname{int}(\mathcal{H})$.

Therefore, if case (2) is to hold, we assume that for certain $j=j^*$ we have

$$\operatorname{dist}\bigl(\xi_{j*}^n,\partial\Omega_{\delta_n}\bigr)=0.$$

Assume first that there exists a constant C such that $\delta_n\leqslant C\rho_n$. Consider the following sum (summation here is taken with respect to all $i\neq j$):

$$s_n=\sum_{i\neq j}G\bigl(\xi_j^n,\xi_i^n\bigr).$$

The leading part, as $n\to\infty$, of s_n comes just from the points that become close as $n\to\ 0$. We can isolate groups of those points according to the asymptotic form of their mutual distances. For example we can define:

$$\rho_n^1=\inf_{i\neq j,\ i,j>k}\bigl|\xi_j^n-\xi_i^n\bigr|,$$

and consider those points whose mutual distances are $O(\rho_n^1)$, and so on. For each group of those points (also those with indices higher than k) the argument given above in the case (1) applies. This means that not only those points become close to one another but also that their distance to the boundary $\partial\Omega_{\delta_n}$ is comparable with their mutual distance. Applying the asymptotic formula for the Green's function we see that

$$s_n=O(1),\quad \text{as } n\to\infty. \tag{8.10}$$

Since $|\xi_{j^*}^n - \bar{\xi}_{j^*}^n| \leqslant 2\delta_n$ (because $\xi_{j^*}^n \in \partial\Omega_{\delta_n}$) we have that

$$\sum_j H(\xi_j^n, \xi_j^n) \to \infty, \quad \text{as } n \to \infty,$$

which together with (8.10) contradicts the fact that $W_{\mathcal{N}}(\xi^n, \mathbb{1})$ is bounded uniformly in n.

Finally assume that $\rho_n = o(\delta_n)$. In this case after scaling with ρ_n around $\xi_{j^*}^n$ and arguing similarly as in the case (1) we get a contradiction with (8.8), (8.9) since those points ξ_j^n that are on $\partial\Omega_{\delta_n}$, after passing to the limit, give rise to points that lie on the same straight line. Thus case (2) cannot hold.

In summary we reached a contradiction and Claim 2 follows. The proof is complete.

8.3. *Proof of Theorem 1.3*

For part (b), let us consider now $k = d$ and $\mathbb{d} = (1, \dots, 1)$. Topologically, the domain $\mathcal{D}$ of W_g is equivalent to

$$\Omega \times (\Omega \setminus \{P_1\}) \times \cdots \times (\Omega \setminus \{P_1, \dots, P_d\}),$$

where the P_i's are distinct points of Ω. Since W_g approaches in this situation $+\infty$ near the boundary, $h = \mathrm{Cat}(\mathcal{D})$ critical values c_j can be defined as

$$c_j \equiv \inf_{\mathrm{Cat}(A) \geqslant j} \sup_A W_g, \quad A \subset \mathcal{D}.$$

If these values are all distinct, the same is true for any small C^1-perturbation, and at least k critical points, distinct up to permutations of coordinates, are present. If two of them coincide, automatically an infinite number of critical points is present. If Ω is simply connected, h is at least equal to the category of the $(d-1)$-torus, namely $h \geqslant d$. If Ω is not simply connected this category is at least that of the d-torus, namely $d+1$. This concludes the proof.

For part (a) we consider $k = 2$ and $\mathbb{d} = (+1, -1)$. In this case we consider Neumann Green's function, solution of

$$-\Delta_x G(x, \xi) = 2\pi \left(\delta(x - \xi) - \frac{1}{|\Omega|} \right) \quad \text{in } \Omega,$$

$$\frac{\partial G}{\partial \nu_x} = 0 \quad \text{on } \partial\Omega, \qquad \int_\Omega G = 0,$$

and write $H(x, \xi) = G(x, \xi) + \log|x - \xi|$. Then, from representation (6.10) we find that

$$W_g(\xi, \mathbb{d}) = -2\pi G(\xi_1, \xi_2) - \pi H(\xi_1, \xi_1) - \pi H(\xi_2, \xi_2) + \Theta(\xi_1, \xi_2),$$

where Θ and its derivatives are bounded. Asymptotic behavior of this function when points ξ_i are either close to the boundary or to each other is analogous to that of the function $W_{\mathcal{N}}$ in the proof of Theorem 1.2(c) for $m = 2$ except that here it carries opposite sign. The same arguments as in that proof then apply to construct a non-trivial critical point situation for W_g and the desired result follows.

Appendix A

In this appendix we will prove an estimate leading to formula (4.14).

Let us consider the bilinear form associated to the operator L^0 in (3.22) with $\mathrm{w}(x) = U(r)e^{id\theta}$, $d = \pm 1$.

$$B(\phi,\phi) = \int_{\mathbb{R}^2} |\nabla\phi|^2 - \int_{\mathbb{R}^2} \left(1-U^2\right)|\phi|^2 + 2\int_{\mathbb{R}^2} \left|\mathrm{Re}(\bar{\mathrm{w}}\phi)\right|^2,$$

defined in its natural space H of all locally-H^1 functions with

$$\|\phi\|_H = \int_{\mathbb{R}^2} |\nabla\phi|^2 + \int_{\mathbb{R}^2} \left(1-U^2\right)|\phi|^2 + \int_{\mathbb{R}^2} \left|\mathrm{Re}(\bar{\mathrm{w}}\phi)\right|^2 < +\infty.$$

Let us consider, for a given ϕ, its associated ψ defined by the relation

$$\phi = i\mathrm{w}\psi. \tag{A.1}$$

Let us decompose

$$\psi = \psi_0(r) + \sum_{m\geqslant 1}\left[\psi_m^1(x) + \psi_m^2(x)\right], \tag{A.2}$$

where

$$\psi_0 = \psi_{01}(r) + i\psi_{02}(r),$$
$$\psi_m^1 = \psi_{m1}^1(r)\cos(m\theta) + i\psi_{m2}^1\sin(m\theta),$$
$$\psi_m^2 = \psi_{m1}^2(r)\sin(m\theta) + i\psi_{m2}^2\cos(m\theta).$$

This bilinear form is non-negative, as it follows from various results in [3,4,26,27,34], see also [10,29].

We want to prove the following fact.

Lemma A.1. *There exists a constant $C > 0$ such that if $\phi \in H$ decomposes like in* (A.1), (A.2) *with $\psi_0 \equiv 0$, and satisfies the orthogonality conditions*

$$\mathrm{Re}\int_{B(0,1/2)} \mathrm{w}_{x_l}\bar{\phi} = 0, \quad l = 1,2,$$

then

$$B(\phi,\phi) \geqslant C\int_{\mathbb{R}^2} \frac{|\phi|^2}{1+r^2}. \tag{A.3}$$

Proof. Let ϕ be as in the statement of the theorem. We shall establish the above result assuming first that ϕ is smooth, compactly supported and that its support does not contain zero. Then we have the identity

$$B(\phi,\phi) = B(i\mathrm{w}\psi, i\mathrm{w}\psi) = \mathcal{B}_0(\psi,\psi),$$

where, explicitly,

$$\mathcal{B}_0(\psi,\psi) = \int_{\mathbb{R}^2} |\nabla\psi|^2 U^2 + \int_{\mathbb{R}^2} |\psi_2|^2 U^4 - 2\int_{\mathbb{R}^2} \nabla\theta\cdot[\psi_1\nabla\psi_2 - \psi_2\nabla\psi_1]U^2.$$

The function ψ satisfies then the orthogonality conditions

$$\mathrm{Re}\left\{\int_{B(0,1/2)} \psi\left[d\frac{\partial\theta}{\partial x_j} + \frac{i}{U}\frac{\partial U(r)}{\partial x_j}\right]U^2(r)\right\} = 0, \tag{A.4}$$

$j = 1, 2$. It is easy to check that $\mathcal{B}_0$ separates Fourier modes. Since $\psi_0 = 0$ we get

$$\mathcal{B}_0(\psi,\psi) = \sum_{m\geqslant 1}\left[\mathcal{B}_0\left(\psi_m^1,\psi_m^1\right) + \mathcal{B}_0\left(\psi_m^2,\psi_m^2\right)\right].$$

Expressed in terms of ψ_m^j the bilinear forms take form

$$\mathcal{B}_0(\psi,\psi) = \pi\sum_{j=1,2}\sum_{m\geqslant 1}\mathbb{B}_m^j\left(\psi_m^j,\psi_m^j\right),$$

where for a radial $\mathbb{R}^2$-valued function $\varphi = \varphi(r)$ we denote

$$\mathbb{B}_m^j(\varphi,\varphi) = \int_0^\infty |\varphi'|^2 U^2 r\,dr + 2\int_0^\infty \varphi_2^2 U^4 r\,dr + \int_0^\infty B_m^j\varphi\cdot\varphi U^2 r\,dr,$$

where

$$B_m^j = \frac{1}{r^2}\begin{pmatrix} m^2 & 2(-1)^j m \\ 2(-1)^j m & m^2 \end{pmatrix}.$$

Then we need to show that

$$\mathcal{B}_0(\psi,\psi) \geqslant C\sum_{m\geqslant 1}\int_0^\infty\left[|\psi_{m1}|^2 + |\psi_{m2}|^2\right]\frac{U^2(r)r\,dr}{1+r^2} \tag{A.5}$$

under assumption (A.4).

Claim 1. *There exists $C > 0$ such that*

$$\mathbb{B}_1^j(\varphi, \varphi) \geqslant C \int_0^\infty \frac{|\varphi|^2 r\,dr}{1+r^2}, \quad j = 1, 2, \tag{A.6}$$

for each function φ radial and compactly supported which satisfies

$$\int_0^{1/2} \varphi \cdot \mathrm{Z}_0 U^2 r\,dr = 0, \tag{A.7}$$

where

$$\mathrm{Z}_0(r) = (d/r, -U'/U).$$

In [10, Proposition 2.1], it was proven the identity

$$\mathbb{B}_1^j(\varphi, \varphi) = \int_0^\infty \left|\varphi' - A(r)\varphi\right|^2 U^2(r) r\,dr, \tag{A.8}$$

where $A(r)$ is a 2×2 symmetric matrix of functions for which a function φ satisfies

$$\varphi' = A(r)\varphi, \qquad \int_0^\infty |\varphi|^2 U^2 r\,dr < +\infty,$$

if and only if φ is a constant multiple of the function $\mathrm{Z}_0(r)$.

Next, let us notice that for a sufficiently large R and M we have

$$\int_0^R M\varphi_2^2 r\,dr + \int_0^R \left(|\varphi_1|^2 - 4\varphi_1\varphi_2 + |\varphi_2|^2\right)\frac{U^2}{r^2} r\,dr \geqslant c \int_0^R |\varphi|^2 r\,dr,$$

with certain constant $c > 0$. It then follows that there exist constants $c_1, c_2 > 0$ such that

$$\mathbb{B}_1^j(\varphi, \varphi) \geqslant c_1 \int_0^\infty |\varphi|^2 \frac{r\,dr}{1+r^2} - c_2 \int_0^R \varphi_2^2 r\,dr. \tag{A.9}$$

Now, if (A.6) were not true then for a sequence of $\varepsilon_n \to 0$ and functions φ^n satisfying (A.7) we would have

$$\int_0^\infty \left|\varphi^n\right|^2 \frac{r\,dr}{1+r^2} = 1,$$

while simultaneously $\mathbb{B}_1^j(\varphi^n, \varphi^n) \to 0$. Estimate (A.9) additionally implies that if we set

$$\|\varphi\|_{H^*}^2 \equiv \int_0^\infty \left(|\varphi'|^2 + \frac{|\varphi|^2}{r^2} + U^2|\varphi_2|^2 \right) U^2 r\, dr,$$

then φ_n is bounded in this space. In particular we may assume the this sequence has a weak limit $\tilde{\varphi} \in H^*$. Moreover, this function satisfies, thanks to (A.8), $\tilde{\varphi}' = A(r)\tilde{\varphi}$ and hence $\tilde{\varphi} = C_0 \mathrm{Z}_0$. But also

$$\int_0^{1/2} \tilde{\varphi} \cdot \mathrm{Z}_0 U^2 r\, dr = 0 \tag{A.10}$$

and then $\tilde{\varphi} = 0$. But from (A.9) we infer that $\tilde{\varphi} \neq 0$. We have reached a contradiction that completes the proof of the claim.

Claim 2. *For each $m > 1$ we have*

$$\mathbb{B}_m^j(\varphi, \varphi) \geqslant C \int_0^\infty \frac{|\varphi|^2 r\, dr}{1 + r^2}, \quad j = 1, 2, \tag{A.11}$$

for each radial function φ.

To prove this we just observe that

$$\left(B_m^j - B_1^j\right)\varphi \cdot \varphi \geqslant \frac{(m-1)^2}{r^2}|\varphi|^2,$$

hence, using $\mathbb{B}_1^j(\varphi, \varphi) \geqslant 0$, one proves the claim.

Going back to the proof of (A.3) we see that from the above claims we readily obtain (A.5). To establish the final result, lifting the requirement that ϕ vanishes near the origin we argue by approximation using a shrinking sequence of cut-off functions, as similarly done in [10]. This concludes the proof. □

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